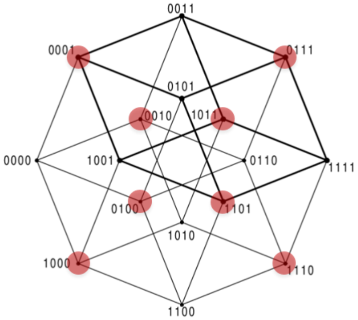
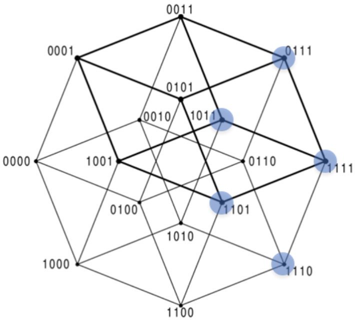
Parity Helps to Compute Majority



Igor Carboni Oliveira

Rahul Santhanam

Srikanth Srinivasan



Computational Complexity Conference 2019

Background and Motivation

Bounded-depth boolean circuits

- AC^0: Bounded-depth circuits with **AND**, **OR**, **NOT** gates.
- A model that captures **fast parallel computations**.
- Close connections to **logic** and **finite model theory**.

We know a lot about AC^0

- Explicit lower bounds: $2^{\Omega(n^{1/(d-1)})}$ for Parity_n and $\mathsf{Majority}_n$.

- Lower bound techniques have led to several advances:

  – Learning Algorithms for AC^0 using random examples.

  – PRGs for AC^0 with poly-log seed length.

  – Exponential lower bounds for AC^0-Frege.

We know a lot about AC^0

- Explicit lower bounds: $2^{\Omega(n^{1/(d-1)})}$ for Parity_n and $\mathsf{Majority}_n$.

- Lower bound techniques have led to several advances:

– Learning Algorithms for AC^0 using random examples.

– PRGs for AC^0 with poly-log seed length.

– Exponential lower bounds for AC^0-Frege.

This talk: $\mathsf{AC}^0[\oplus]$ circuits

- $\mathsf{AC}^0[\oplus]$: Extension of AC^0 by $\oplus$ (parity) gates.

- Parities can be **very helpful**: error-correcting codes, hash functions, $\mathsf{GF}(2)$-polynomials, combinatorial designs, $\ldots$

This talk: $\mathsf{AC}^0[\oplus]$ circuits

- $\mathsf{AC}^0[\oplus]$: Extension of AC^0 by $\oplus$ (parity) gates.

- Parities can be **very helpful**: error-correcting codes, hash functions, $\mathrm{GF}(2)$-polynomials, combinatorial designs, ...

- Explicit lower bounds: $2^{\Omega(n^{1/2(d-1)})}$ for $\mathrm{Majority}_n$.

This talk: $\mathsf{AC}^0[\oplus]$ circuits

- $\mathsf{AC}^0[\oplus]$: Extension of AC^0 by $\oplus$ (parity) gates.

- Parities can be **very helpful**: error-correcting codes, hash functions, GF(2)-polynomials, combinatorial designs, ...

- Explicit lower bounds: $2^{\Omega(n^{1/2(d-1)})}$ for Majority$_n$.

- AC^0 and $\mathsf{AC}^0[\oplus]$ are significantly different circuit classes: **Example:** depth hierarchy for AC^0, depth collapse for $\mathsf{AC}^0[\oplus]$.

- Many fundamental questions remain wide open for $\mathrm{AC}^0[\oplus]$.

– Can we learn $\mathrm{AC}^0[\oplus]$ using random examples?

– Are there PRGs of seed length $o(n)$?

– Does every tautology admit a short $\mathrm{AC}^0[\oplus]$-Frege proof?

AC^0 **versus** $\mathsf{AC}^0[\oplus]$

- Our primitive understanding of $\mathsf{AC}^0[\oplus]$ is reflected in part on existing lower bounds:

– Majority is one of the most studied boolean functions.

– Depth-d AC^0 complexity of Majority is $2^{\widetilde{\Theta}(n^{1/(d-1)})}$ (1980's).

– Best known $\mathsf{AC}^0[\oplus]$ lower bound is $2^{\Omega(n^{1/2(d-1)})}$ for any $f \in \mathsf{NP}$.

(Razborov-Smolensky approximation method, 1980's)

Question. Can $\oplus$ gates help us computing Majority?

- Our primitive understanding of $\mathsf{AC}^0[\oplus]$ is reflected in part on existing lower bounds:

  - Majority is one of the most studied boolean functions.
  - Depth-d AC^0 complexity of Majority is $2^{\widetilde{\Theta}(n^{1/(d-1)})}$ (1980's).
  - Best known $\mathsf{AC}^0[\oplus]$ lower bound is $2^{\Omega(n^{1/2(d-1)})}$ for any $f \in \mathsf{NP}$.

(Razborov-Smolensky approximation method, 1980's)

Question. Can $\oplus$ gates help us computing Majority?

AC^0 **versus** $\mathsf{AC}^0[\oplus]$

- Our primitive understanding of $\mathsf{AC}^0[\oplus]$ is reflected in part on existing lower bounds:

– Majority is one of the most studied boolean functions.

– Depth-d AC^0 complexity of Majority is $2^{\widetilde{\Theta}(n^{1/(d-1)})}$ (1980's).

– Best known $\mathsf{AC}^0[\oplus]$ lower bound is $2^{\Omega(n^{1/2(d-1)})}$ for any $f \in \mathsf{NP}$.

(Razborov-Smolensky approximation method, 1980's)

Question. Can $\oplus$ gates help us computing Majority?

AC^0 **versus** $\mathsf{AC}^0[\oplus]$

- Our primitive understanding of $\mathsf{AC}^0[\oplus]$ is reflected in part on existing lower bounds:

– $\mathsf{Majority}$ is one of the most studied boolean functions.

– Depth-d AC^0 complexity of $\mathsf{Majority}$ is $2^{\widetilde{\Theta}(n^{1/(d-1)})}$ (1980's).

– Best known $\mathsf{AC}^0[\oplus]$ lower bound is $2^{\Omega(n^{1/2(d-1)})}$ for any $f \in \mathsf{NP}$.

(Razborov-Smolensky approximation method, 1980's)

Question. Can $\oplus$ gates help us computing $\mathsf{Majority}$?

Why should we care?

1. Combinatorics: huge gap between $2^{n^{1/(d-1)}}$ and $2^{n^{1/2(d-1)}}$.

Why should we care?

1. Combinatorics: huge gap between $2^{n^{1/(d-1)}}$ and $2^{n^{1/2(d-1)}}$.

2. Can we beat the "obviously" optimal algorithm?

Why should we care?

1. Combinatorics: huge gap between $2^{n^{1/(d-1)}}$ and $2^{n^{1/2(d-1)}}$.

2. Can we beat the “obviously” optimal algorithm?

3. Parity gates play crucial role in hardness magnification.

Example: “a layer of parities away from NC^1 lower bounds”.

Why should we care?

1. Combinatorics: huge gap between $2^{n^{1/(d-1)}}$ and $2^{n^{1/2(d-1)}}$.

2. Can we beat the "obviously" optimal algorithm?

3. Parity gates play crucial role in hardness magnification.

Example: "a layer of parities away from NC^1 lower bounds".

4. Better understanding of circuit complexity of a class $\mathcal{C}$ often leads to progress w.r.t. related questions.

Results

Informal Summary

- Neither the trivial upper bound of $2^{\widetilde{O}(n^{1/(d-1)})}$ gates nor the Razborov-Smolensky lower bound $2^{\Omega(n^{1/2(d-1)})}$ is tight.

Our new upper and lower bounds for $\mathsf{AC}^0[\oplus]$ show that:

- Parity gates can speedup the computation of Majority for each large depth $d \in \mathbb{N}$.

- Indeed, the AC^0 and $\mathsf{AC}^0[\oplus]$ complexities are similar at depth 3, but parity gates significantly help at depth 4.

Informal Summary

- Neither the trivial upper bound of $2^{\widetilde{O}(n^{1/(d-1)})}$ gates nor the Razborov-Smolensky lower bound $2^{\Omega(n^{1/2(d-1)})}$ is tight.

Our new upper and lower bounds for $\mathsf{AC}^0[\oplus]$ **show that:**

- Parity gates can speedup the computation of $\mathsf{Majority}$ for each large depth $d \in \mathbb{N}$.
- Indeed, the AC^0 and $\mathsf{AC}^0[\oplus]$ complexities are similar at depth 3, but parity gates significantly help at depth 4.

Divide-and-conquer is not optimal for $\mathsf{AC}^0[\oplus]$

Recall: For $d \geq 2$, the depth-d AC^0 complexity of $\mathsf{Majority}_n$ is $2^{\widetilde{\Theta}\left(n^{1/(d-1)}\right)}$.

Theorem 1. Let $d \geq 5$ be an integer. Majority on n bits can be computed by depth-d $\mathsf{AC}^0[\oplus]$ circuits of size $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$.

- A similar upper bound holds for **symmetric functions** and **linear threshold functions.**

Recall: For $d \geq 2$, the depth-d AC^0 complexity of $\mathsf{Majority}_n$ is $2^{\widetilde{\Theta}\left(n^{1/(d-1)}\right)}$.

Theorem 1. Let $d \geq 5$ be an integer. Majority on n bits can be computed by depth-d $\mathsf{AC}^0[\oplus]$ circuits of size $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$.

- A similar upper bound holds for **symmetric functions** and **linear threshold functions.**

Strengthening Razborov-Smolensky

Razborov-Smolensky

The depth-d $\mathrm{AC}^0[\oplus]$ complexity of $\mathsf{Majority}_n$ is $2^{\Omega\left(n^{1/(2d-2)}\right)}$.

Theorem 2. Let $d \geq 3$ be an integer. Majority on n bits requires depth-d $\mathrm{AC}^0[\oplus]$ circuits of size $2^{\Omega\left(n^{1/(2d-4)}\right)}$.

- A small improvement of explicit lower bounds for $f \in \mathrm{NP}$.
- This improvement is significant for very small d.

Strengthening Razborov-Smolensky

Razborov-Smolensky

The depth-d $\mathsf{AC}^0[\oplus]$ complexity of $\mathsf{Majority}_n$ is $2^{\Omega\left(n^{1/(2d-2)}\right)}$.

Theorem 2. Let $d \geq 3$ be an integer. Majority on n bits requires depth-d $\mathsf{AC}^0[\oplus]$ circuits of size $2^{\Omega\left(n^{1/(2d-4)}\right)}$.

- A small improvement of explicit lower bounds for $f \in \mathsf{NP}$.
- This improvement is significant for very small d.

New lower bound + extension of upper bound techniques yield:

Corollary 1.

The depth-3 $\mathrm{AC}^0[\oplus]$ circuit size complexity of Majority is $2^{\widetilde{\Theta}(n^{1/2})}$.

The depth-4 $\mathrm{AC}^0[\oplus]$ circuit size complexity of Majority is $2^{\widetilde{\Theta}(n^{1/4})}$.

- Parity gates significantly help at depth 4 but not at depth 3.

Techniques: $\mathsf{AC}^0[\oplus]$ Upper Bounds

Improved upper bound for all large depths

Theorem 1. Let $d \geq 5$ be an integer. Majority on n bits can be computed by depth-d $\mathsf{AC}^0[\oplus]$ circuits of size $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$.

$$E_i(y) = \begin{cases} 1 & \text{if } \ |y|_1 = i, \\ 0 & \text{otherwise.} \end{cases}$$

$$D_{i,j}(y) = \begin{cases} 1 & \text{if } \ |y|_1 = i, \\ 0 & \text{if } \ |y|_1 = j. \end{cases}$$

Goal: $\mathsf{AC}^0[\oplus]$ circuits of size $\approx 2^{n^{2/3d}}$ for all $D_{i,j}$, $0 \leq i \neq j \leq n$.

Theorem 1. Let $d \geq 5$ be an integer. Majority on n bits can be computed by depth-d $\mathrm{AC}^0[\oplus]$ circuits of size $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$.

$$E_i(y) = \begin{cases} 1 & \text{if } \ |y|_1 = i, \\ 0 & \text{otherwise.} \end{cases} \qquad D_{i,j}(y) = \begin{cases} 1 & \text{if } \ |y|_1 = i, \\ 0 & \text{if } \ |y|_1 = j. \end{cases}$$

Goal: $\mathrm{AC}^0[\oplus]$ circuits of size $\approx 2^{n^{2/3d}}$ for all $D_{i,j}$, $0 \leq i \neq j \leq n$.

Improved upper bound for all large depths

Theorem 1. Let $d \geq 5$ be an integer. Majority on n bits can be computed by depth-d $\mathrm{AC}^0[\oplus]$ circuits of size $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$.

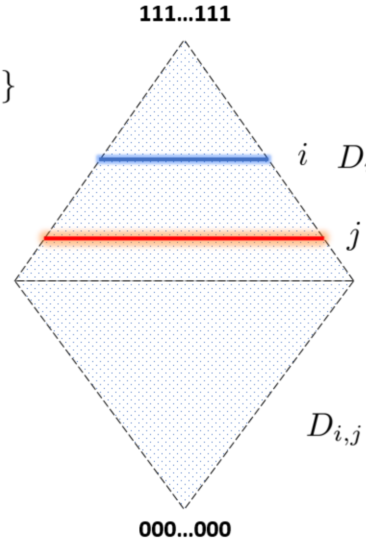
$$E_i(y) = \begin{cases} 1 & \text{if } |y|_1 = i, \\ 0 & \text{otherwise.} \end{cases} \qquad D_{i,j}(y) = \begin{cases} 1 & \text{if } |y|_1 = i, \\ 0 & \text{if } |y|_1 = j. \end{cases}$$

Goal: $\mathrm{AC}^0[\oplus]$ circuits of size $\approx 2^{n^{2/3d}}$ for all $D_{i,j}$, $0 \leq i \neq j \leq n$.

The $D_{i,j}$ partial boolean function

$D_{i,j} : \{0,1\}^n \to \{0,1\}$

$i, j \in [n]$



$D_{i,j}(x) = 1$

$D_{i,j}(x) = 0$

$D_{i,j}(x) \in \{0,1\}$

Circuit for $D_{i,j}$

- We consider the value $|i - j|$:

– **Small regime:** $|i - j| \leq n^{1/3}$.

We use an "**algebraic**" construction. This circuit relies on a $\mathbb{F}_2$ polynomial, divide-and-conquer, and needs $\oplus$ gates.

– **Large regime:** $|i - j| > n^{1/3}$.

We use a "**combinatorial**" construction. This circuit relies on a probabilistic construction of AC^0 circuits for the *Coin Problem*.

Circuit for $D_{i,j}$

- We consider the value $|i - j|$:

– **Small regime:** $|i - j| \leq n^{1/3}$.

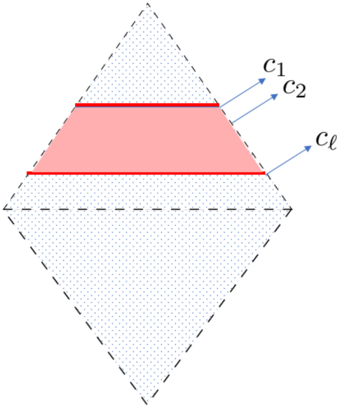
We use an "**algebraic**" construction. This circuit relies on a $\mathbb{F}_2$ polynomial, divide-and-conquer, and needs $\oplus$ gates.

– **Large regime:** $|i - j| > n^{1/3}$.

We use a "**combinatorial**" construction. This circuit relies on a probabilistic construction of AC^0 circuits for the *Coin Problem*.

$|i-j| \leq n^{1/3}$: The algebraic construction I

Lemma [AW15]:



$c_1, c_2, \ldots, c_\ell \in \mathbb{Z}$

There is a polynomial $Q\colon \{0,1\}^n \to \mathbb{Z}$ such that:

$Q(x) = c_i$ when $|x|_1$ agrees with corresponding layer.

Moreover,

$\deg(Q) \leq \ell - 1$ and $Q(x) = \sum_{t=0}^{\ell-1} a_t \cdot Q_t(x)$

$a_t \in \mathbb{Z}$

$$Q_t(x) = \sum_{S \in \binom{[n]}{t}} \prod_{j \in S} x_j$$

t-th symmetric elementary polynomial

$|i-j| \leq n^{1/3}$: The algebraic construction II

- $Q(x_1, \ldots, x_n)$ is defined over $\mathbb{Z}$. We take a homomorphism $\psi \colon \mathbb{Z} \to \mathbb{F}_2$.

$$P(x) = \sum_{t=0}^{\ell-1} b_t \cdot P_t(x) \text{ over } \mathbb{F}_2, \text{ where } \ell = (i-j)+1.$$

- $P(x)$ computes $D_{i,j}(x)$ and has degree at most $\ell \leq n^{1/3}$.

– We would like to compute $P(x)$ in depth-d $\mathsf{AC}^0[\oplus]$.

– **Goal:** elementary symmetric polynomials $Q_1, \ldots, Q_\ell$, where $\ell \leq n^{1/3}$.

$|i-j| \le n^{1/3}$: The algebraic construction II

- $Q(x_1, \ldots, x_n)$ is defined over $\mathbb{Z}$. We take a homomorphism $\psi\colon \mathbb{Z} \to \mathbb{F}_2$.

$$P(x) = \sum_{t=0}^{\ell-1} b_t \cdot P_t(x) \ \text{ over } \mathbb{F}_2, \ \text{ where } \ \ell = (i-j)+1.$$

- $P(x)$ computes $D_{i,j}(x)$ and has degree at most $\ell \le n^{1/3}$.

– We would like to compute $P(x)$ in depth-d AC$^0[\oplus]$.

– **Goal:** elementary symmetric polynomials $Q_1, \ldots, Q_\ell$, where $\ell \le n^{1/3}$.

$|i-j| \leq n^{1/3}$: The algebraic construction II

- $Q(x_1, \ldots, x_n)$ is defined over $\mathbb{Z}$. We take a homomorphism $\psi\colon \mathbb{Z} \to \mathbb{F}_2$.

$$P(x) = \sum_{t=0}^{\ell-1} b_t \cdot P_t(x) \text{ over } \mathbb{F}_2, \text{ where } \ell = (i-j)+1.$$

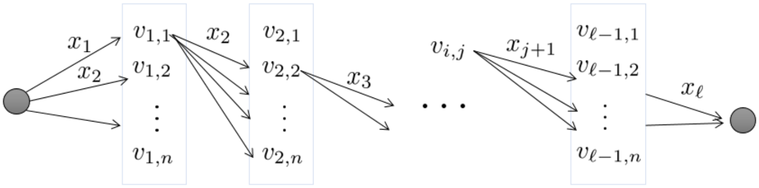
- $P(x)$ computes $D_{i,j}(x)$ and has degree at most $\ell \leq n^{1/3}$.

– We would like to compute $P(x)$ in depth-d AC$^0[\oplus]$.

– **Goal:** elementary symmetric polynomials $Q_1, \ldots, Q_\ell$, where $\ell \leq n^{1/3}$.

$$P_\ell(x_1,\ldots,x_n) = \sum_{S\in\binom{[n]}{\ell}} \prod_{j\in S} x_j$$

We simulate P_ℓ using an **algebraic branching program**:



$\Sigma\Pi$
width n
length $\ell+1$

Divide-and-conquer approach similar to depth-d circuit for STCONN:

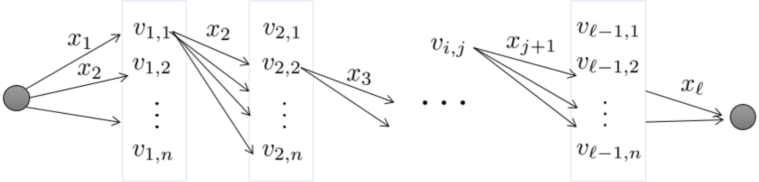
We can compute P_ℓ using $\bigwedge$ and $\bigoplus$ in depth d and size $n^{O(\ell^{2/d})}$.

For $\ell \le n^{1/3}$, this gives $\mathrm{AC}^0[\oplus]$ circuit size $2^{\tilde{O}(n^{2/3d})}$.

$|i-j| \le n^{1/3}$: The algebraic construction III

$$P_\ell(x_1,\ldots,x_n) = \sum_{S \in \binom{[n]}{\ell}} \prod_{j \in S} x_j$$

We simulate P_ℓ using an **algebraic branching program**:



$\Sigma\Pi$
width n
length $\ell + 1$

Divide-and-conquer approach similar to depth-d circuit for STCONN:

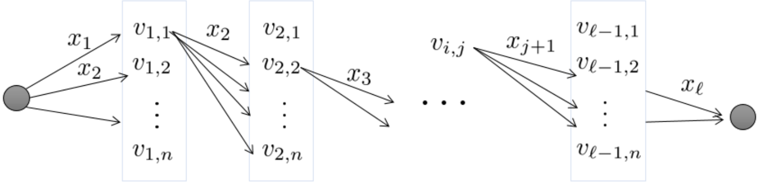
We can compute P_ℓ using $\bigwedge$ and $\bigoplus$ in depth d and size $n^{O(\ell^{2/d})}$.

For $\ell \le n^{1/3}$, this gives $AC^0[\oplus]$ circuit size $2^{\tilde{O}(n^{2/3d})}$.

$|i-j| \leq n^{1/3}$: The algebraic construction III

$$P_\ell(x_1, \ldots, x_n) = \sum_{S \in \binom{[n]}{\ell}} \prod_{j \in S} x_j$$

We simulate P_ℓ using an **algebraic branching program**:



$\Sigma\Pi$
width n
length $\ell + 1$

Divide-and-conquer approach similar to depth-d circuit for STCONN:

We can compute P_ℓ using $\bigwedge$ and $\bigoplus$ in depth d and size $n^{O(\ell^{2/d})}$.

For $\ell \leq n^{1/3}$, this gives $\mathrm{AC}^0[\oplus]$ circuit size $2^{\tilde{O}(n^{2/3d})}$.

$|i-j| > n^{1/3}$: The combinatorial construction

By moving from n to $\Theta(n)$ input bits, we can assume i and j are equally spaced from middle layer.

Let $i = n/2 + t$ and $j = n/2 - t$. Enough to compute *Approximate Majority / Coin Problem.*

Elegant construction [OW07], [Ama09], [RS17]:
Can be done by depth-d AC^0 circuits of size roughly $2^{(n/t)^{1/d}}$.

For $t = \Theta(|i-j|) > n^{1/3}$, this size bound is $2^{O(n^{2/3d})}$.

$|i-j| > n^{1/3}$: The combinatorial construction

By moving from n to $\Theta(n)$ input bits, we can assume i and j are equally spaced from middle layer.

Let $i = n/2 + t$ and $j = n/2 - t$. Enough to compute *Approximate Majority* / *Coin Problem*.

Elegant construction [OW07], [Ama09], [RS17]:
Can be done by depth-d AC^0 circuits of size roughly $2^{(n/t)^{1/d}}$.

For $t = \Theta(|i-j|) > n^{1/3}$, this size bound is $2^{O(n^{2/3d})}$.

$|i-j| > n^{1/3}$: The combinatorial construction

By moving from n to $\Theta(n)$ input bits, we can assume i and j are equally spaced from middle layer.

Let $i = n/2 + t$ and $j = n/2 - t$. Enough to compute *Approximate Majority / Coin Problem*.

Elegant construction [OW07], [Ama09], [RS17]:
Can be done by depth-d AC^0 circuits of size roughly $2^{(n/t)^{1/d}}$.

For $t = \Theta(|i-j|) > n^{1/3}$, this size bound is $2^{O(n^{2/3d})}$.

$|i-j| > n^{1/3}$: The combinatorial construction

By moving from n to $\Theta(n)$ input bits, we can assume i and j are equally spaced from middle layer.

Let $i = n/2 + t$ and $j = n/2 - t$. Enough to compute *Approximate Majority / Coin Problem*.

Elegant construction [OW07], [Ama09], [RS17]:
Can be done by depth-d AC^0 circuits of size roughly $2^{(n/t)^{1/d}}$.

For $t = \Theta(|i-j|) > n^{1/3}$, this size bound is $2^{O(n^{2/3d})}$.

Extensions of the Upper Bound

- Previous argument works for all **symmetric functions**.

- In depth $d = 4$, careful depth control + new ingredient:
randomly splitting variables into buckets.

- **Linear Threshold Functions** (LTFs) and **Polytopes**:
AC^0 reduction to Exact Threshold Functions (ETH) via **[HP10]**,
then reduction to symmetric functions (Chinese remaindering).

Extensions of the Upper Bound

- Previous argument works for all **symmetric functions**.

- In depth $d = 4$, careful depth control + new ingredient: *randomly splitting variables into buckets*.

- **Linear Threshold Functions** (LTFs) and **Polytopes**: AC^0 reduction to Exact Threshold Functions (ETH) via **[HP10]**, then reduction to symmetric functions (Chinese remaindering).

Extensions of the Upper Bound

- Previous argument works for all **symmetric functions**.
- In depth $d = 4$, careful depth control + new ingredient: *randomly splitting variables into buckets*.
- **Linear Threshold Functions** (LTFs) and **Polytopes**: AC^0 reduction to Exact Threshold Functions (ETH) via **[HP10]**, then reduction to symmetric functions (Chinese remaindering).

Techniques: $\mathrm{AC}^0[\oplus]$ Lower Bounds

Theorem 2. Let $d \geq 3$ be an integer. Majority on n bits requires depth-d $\mathrm{AC}^0[\oplus]$ circuits of size $2^{\Omega\left(n^{1/(2d-4)}\right)}$.

Recall: Razborov-Smolensky shows a $2^{\Omega(n^{1/(2d-2)})}$ lower bound.

- Intuition: How to save **two layers of gates** in the polynomial approximation method?

- **Degree Upper Bound:**

Probabilistic polynomial $\boldsymbol{P}$ over $\mathbb{F}_2$ correct on each input w.h.p.

AND, OR, NOT, PARITY: error ε and degree $\log(1/\varepsilon)$

Size-s depth-d $\mathrm{AC}^0[\oplus]$: $\deg(\boldsymbol{P}) \approx (\log s)^{d-1}$ and error $\varepsilon \leq 1/50$.

- **Degree Lower Bound:**

For Majority$_n$, $\deg(\boldsymbol{P})$ must be $\geq \sqrt{n \cdot \log(1/\varepsilon)}$.

- **Degree Upper Bound:**

Probabilistic polynomial $\boldsymbol{P}$ over $\mathbb{F}_2$ correct on each input w.h.p.

AND, OR, NOT, PARITY: error ε and degree $\log(1/\varepsilon)$

Size-s depth-d AC$^0[\oplus]$: deg$(\boldsymbol{P}) \approx (\log s)^{d-1}$ and error $\varepsilon \leq 1/50$.

- **Degree Lower Bound:**

For Majority$_n$, deg$(\boldsymbol{P})$ must be $\geq \sqrt{n \cdot \log(1/\varepsilon)}$.

The lower bound

Putting together the approximate degree bounds:

$$(\log s)^{d-1} \;\geq\; \sqrt{n \cdot \log(1/\varepsilon)}, \qquad \varepsilon = 1/50.$$

This implies that $s \geq 2^{\Omega(n^{1/(2d-2)})}$.

(The RS lower bound is maximized when $\varepsilon =$ constant.)

Our approach

We follow Razborov-Smolensky, with **two new ideas**.

Idea 1. Exploit error $\varepsilon = 1/50$ of polynomial approximator:

– Error is **one-sided** and $\leq 1/\log s$ on say $C^{-1}(1)$.

– Hope to exploit stronger degree lower bound of $\sqrt{n \cdot \log(1/\varepsilon)}$.

Idea 2. Random restrictions for $\mathrm{AC}^0[\oplus]$ circuits:

– Prove that w.h.p. a random restriction leads to depth-2 subcircuits of smaller approximate degree. Can do better than $(\log s)^2$ on bottom layers.

Our approach

We follow Razborov-Smolensky, with **two new ideas**.

Idea 1. Exploit error $\varepsilon = 1/50$ of polynomial approximator:

– Error is **one-sided** and $\leq 1/\log s$ on say $C^{-1}(1)$.

– Hope to exploit stronger degree lower bound of $\sqrt{n \cdot \log(1/\varepsilon)}$.

Idea 2. Random restrictions for $\mathrm{AC}^0[\oplus]$ circuits:

– Prove that w.h.p. a random restriction leads to depth-2 subcircuits of smaller approximate degree. Can do better than $(\log s)^2$ on bottom layers.

First idea: One-sided approximations

- We approximate every non-output gate to error $\leq 1/s^2$.
- By union bound, every input wire of output gate is correct (except with prob. $\leq 1/s$).
- Approximation method over OR gate is one-sided (“random parities”): zero inputs to OR gate always produce zero.

First idea: Stronger degree lower bound

- Smolensky's approximate degree lower bound:

$$\deg_\varepsilon(\text{Majority}_n) = \Omega(\sqrt{n \cdot \log(1/\varepsilon)}).$$

Can we maintain this lower bound when error on $\text{Majority}_n^{-1}(0)$ is $\leq \varepsilon$ but error on $\text{Majority}_n^{-1}(1)$ is as large as $1/50$?

- We extend the technique of **certifying polynomials [KS12]** to show this is the case.

First idea: Stronger degree lower bound

- Smolensky's approximate degree lower bound:

$$\deg_\varepsilon(\mathrm{Majority}_n) = \Omega(\sqrt{n \cdot \log(1/\varepsilon)}).$$

Can we maintain this lower bound when error on $\mathrm{Majority}_n^{-1}(0)$ is $\leq \varepsilon$ but error on $\mathrm{Majority}_n^{-1}(1)$ is as large as $1/50$?

- We extend the technique of **certifying polynomials** **[KS12]** to show this is the case.

Second idea: random restrictions for $\mathsf{AC}^0[\oplus]$

- We prove the following lemma:

Random Restriction Lemma. Let C be a depth-2 $\mathsf{AC}^0[\oplus]$ circuit on n vars and of size $s \geq n^2$. Let $p_* \leq 1/(500 \log s)$. Then,

$$\mathbb{P}_{\boldsymbol{\rho} \sim \mathcal{R}^n_{p*}}\left[\deg_{\varepsilon = 1/s^2}(C|_{\boldsymbol{\rho}}) > 10 \log s \mid \boldsymbol{\rho} \text{ is balanced}\right] < \frac{1}{10s}.$$

- Case analysis based on gates of C (OR, AND, PARITY).

Second idea: random restrictions for $\mathsf{AC}^0[\oplus]$

- We prove the following lemma:

Random Restriction Lemma. Let C be a depth-2 $\mathsf{AC}^0[\oplus]$ circuit on n vars and of size $s \geq n^2$. Let $p_* \leq 1/(500 \log s)$. Then,

$$\mathbb{P}_{\boldsymbol{\rho} \sim \mathcal{R}^n_{p*}} \left[\deg_{\varepsilon = 1/s^2}(C|_{\boldsymbol{\rho}}) > 10 \log s \mid \boldsymbol{\rho} \text{ is balanced} \right] < \frac{1}{10s}.$$

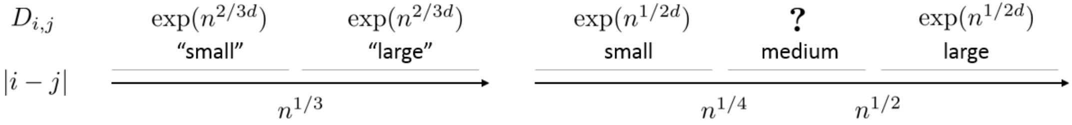
- Case analysis based on gates of C (OR, AND, PARITY).

Concluding Remarks

Open Problems

Challenge: What is the $\mathrm{AC}^0[\oplus]$ complexity of Majority?

- Close the gap between the $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$ upper bound and the $2^{\Omega\left(n^{1/(2d-4)}\right)}$ lower bound.

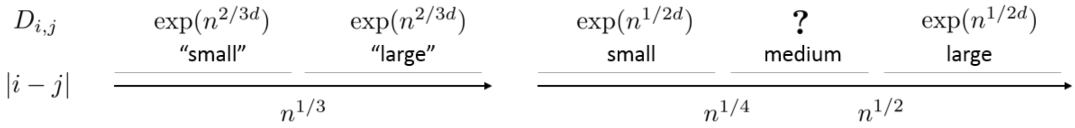


- Find more examples where the "optimal" algorithm or circuit can be improved.

Open Problems

Challenge: What is the $\mathrm{AC}^0[\oplus]$ complexity of Majority?

- Close the gap between the $2^{\widetilde{O}\left(n^{\frac{2}{3}\cdot\frac{1}{(d-4)}}\right)}$ upper bound and the $2^{\Omega\left(n^{1/(2d-4)}\right)}$ lower bound.



- Find more examples where the "optimal" algorithm or circuit can be improved.