Polynomial Identity Testing via Low-Rank Matrices

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Based on work from STOC '22

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Question: Is there an efficient deterministic algorithm for PIT?

HITTING SET GENERATORS

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A map

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$$(y_1, \ldots, y_\ell) \longmapsto (G_1(\bar{y}), \ldots, G_n(\bar{y}))$$

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Parameters:

- seed length ℓ
- degree $\deg(\mathcal{G}) := \max_i \deg(\mathcal{G}_i(\bar{y}))$

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1) Work hard to develop hardness versus randomness
2) Prove lower bounds for $\mathrm{perm}(X)$

Extending Hardness vs Randomness

Can we implement hardness versus randomness for weak circuit classes?

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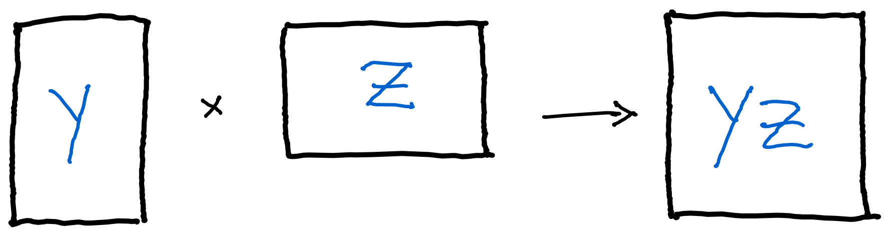
1) is closed under projections
2) ——//—— linear change of variables
3) ——//—— limits
4) requires $n^{\omega(1)}$ size to compute $\det(X)$

THE GENERATOR

Definition: let $r \in \mathbb{N}$ and define

$$G_r : \mathbb{F}^{\sqrt{n} \times r} \times \mathbb{F}^{r \times \sqrt{n}} \longrightarrow \mathbb{F}^{\sqrt{n} \times \sqrt{n}}$$



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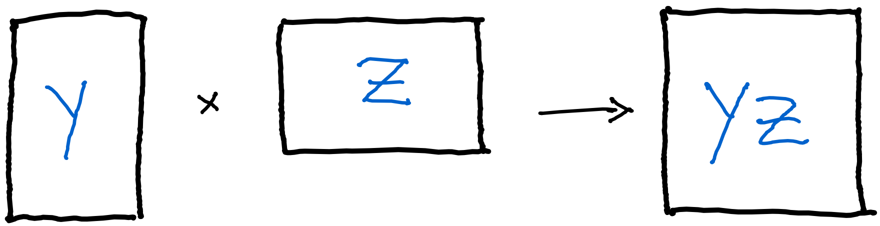

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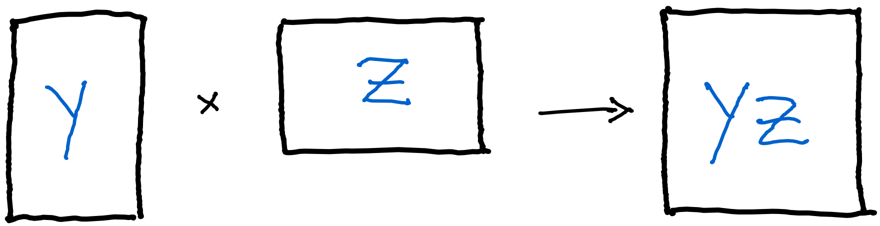

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Lemma [AF]: if a size-s $\mathcal{C}$-circuit vanishes on G_r, then the $r^{1/3} \times r^{1/3}$ determinant has a size-$s^{O(1)}$ $\mathcal{C}$-circuit

Low-Depth Circuits

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Cost: $\Sigma\Pi$ circuit of size $2nr = n s^{o(1)}$ ($n \times r \times n$ matrix multiplication)

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Equivalently, $G_r \circ G_t$ hits $C(\bar{x})$.

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	$\mathcal{G}_r$	$\mathcal{G}_t$	$\mathcal{G}_r \circ \mathcal{G}_t$
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Seed length	$n^{1/2 + o(1)}$	$n^{1/4 + o(1)}$	
Degree	2	2	
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Degree	2	2	4
Cost	$\Sigma\Pi$ ckt $n^{1+o(1)}$ size	$\Sigma\Pi$ ckt $n^{1/2+o(1)}$ size	$\Sigma\Pi\Sigma\Pi$ ckt $n^{1+o(1)}$ size

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Theorem [A-Forbes]: New generators for size-s constant-depth circuits

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Fact

Every generator with seed length ℓ and degree d satisfies

$$\binom{\ell + d}{d} \geq n$$

$$\rightsquigarrow \quad \ell \geq \Omega(n^{1/d})$$

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Cost	General ckt $\tilde{O}(n)$ size	Constant-depth ckt $s^{\Omega(1)}$ size *

VANISHING IDEAL

Given a polynomial map $G : \mathbb{F}^{\ell} \to \mathbb{F}^{n}$,
its vanishing ideal is

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Corollary: Correctness of $\mathcal{G}$ $\equiv$ Circuit lower bounds for $\mathrm{Van}[\mathcal{G}]$

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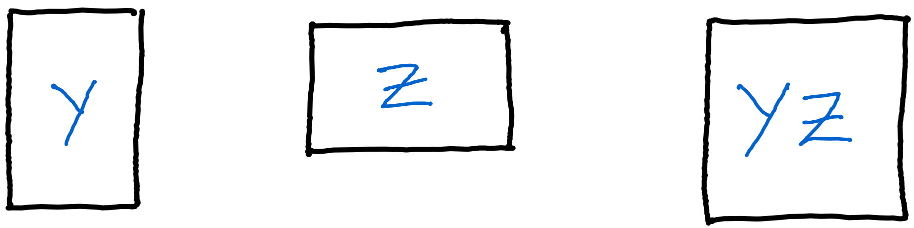

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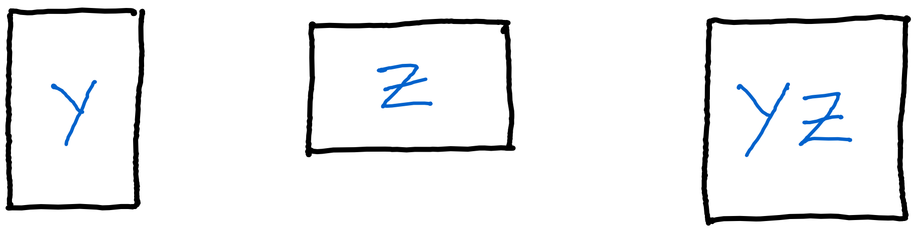

Fact: $\mathrm{Van}[\mathcal{G}_r] = \left\langle \det(X_{A,B}) : \begin{array}{l} A, B \subseteq [n] \\ |A| = |B| = r+1 \end{array} \right\rangle$

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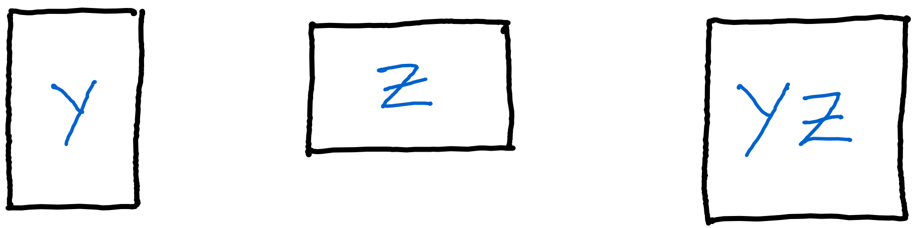

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We want to prove circuit lower bounds for all nonzero polynomials here

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Complexity in Ideals

Goal: small ckt for some $f \in \langle r \times r \text{ minors of } X \rangle$

$\Longrightarrow$ small ckt for $r^{\Theta(1)} \times r^{\Theta(1)}$ determinant

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- For $r < n$, we also have to handle cancellations

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$\forall \varepsilon > 0$, $\forall f(X) \in \langle r \times r$ minors of $n \times n$ matrix $X \rangle$,

Main Technical Result

Theorem [AF '22]

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$\exists$ circuit

Main Technical Result

Theorem [AF '22]

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$\vdots$

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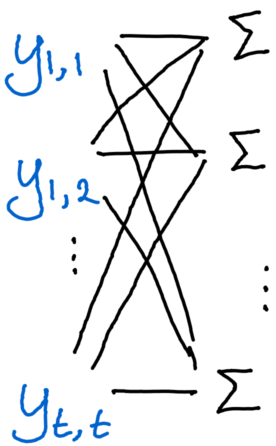

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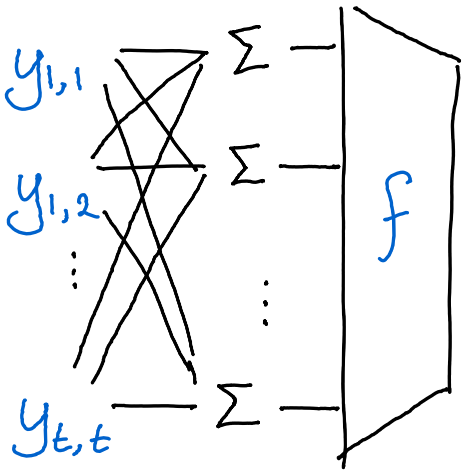

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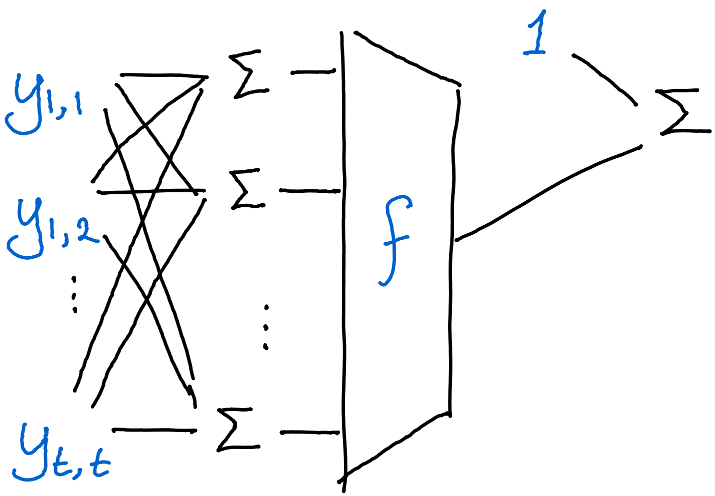

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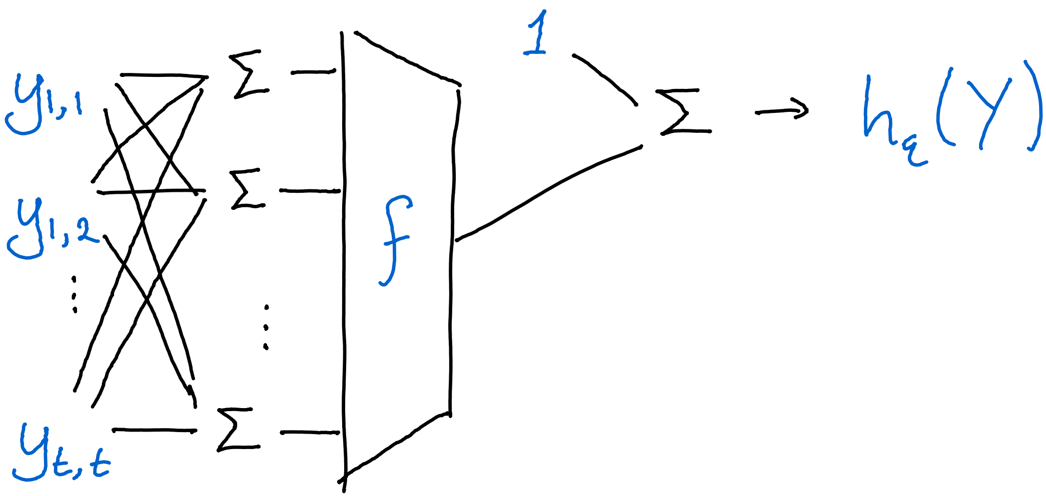

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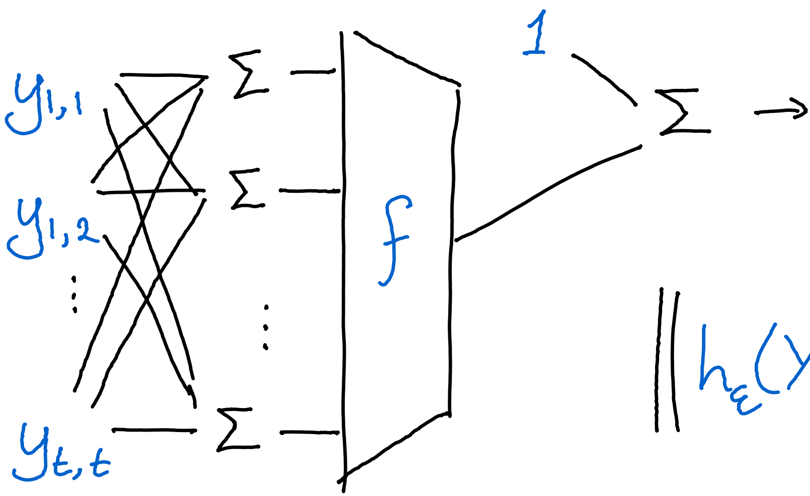

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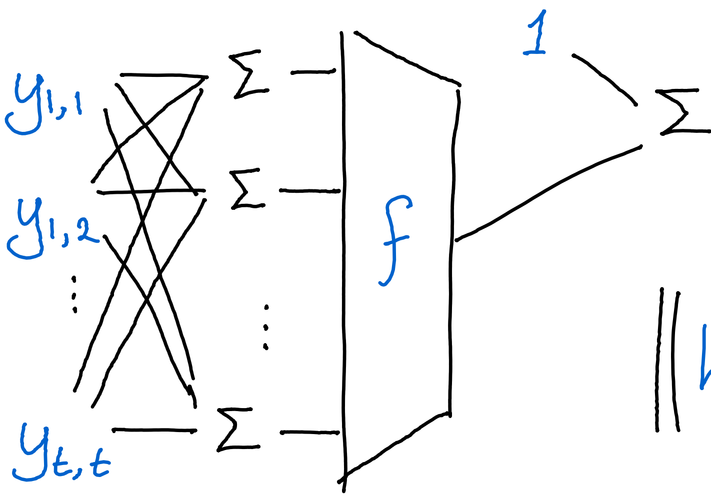

$\rightarrow h_\varepsilon(Y)$

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(if $\mathrm{char}(\mathbb{F}) = p > 0$, $h_\varepsilon \approx \det(Y)^{p^k}$)

PROOF OUTLINE

Given $f(X) \in I := \langle r \times r \text{ minors of } X \rangle$

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1 Linear change of variables $x_{i,j} \longmapsto \ell_{i,j}(X, \varepsilon)$

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2 Well-behaved projection $X \to \mathbb{F}(\varepsilon)[Y]$

sending $h(X) \longrightarrow 1 + \varepsilon \cdot \det_{r^{1/3}}(Y) + O(\varepsilon^2)$

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such that the leading coefficient of $f(L(X,y))$

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Lemma [AF '22]

Let Y be an $m \times m$ matrix of variables.

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Proof uses simulation of ABPs by det [Valiant '79]
+ small ABPs for det [Mahajan-Vinay '97]

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Thank You!