Border rank and homogeneous complexity classes

Fulvio Gesmundo

Saarland University

(based on joint work with P. Dutta, C. Ikenmeyer, G. Jindal, V. Lysikov)

We study complexity measures on complex homogeneous polynomials

$$f \in S^d\mathbb{C}^N = \mathbb{C}[x_1, \dots, x_N]_d.$$

Plan:

- Waring rank and border Waring rank
- Kumar's *product plus constant* model
- Generalization to other complexity classes

Waring rank and border Waring rank

The *Waring rank* of f is

$$\mathrm{WR}(f) = \min\left\{r : f = \ell_1^d + \cdots + \ell_r^d \text{ for some } \ell_j \in S^1\mathbb{C}^n\right\};$$

the *border Waring rank* of f is

$$\underline{\mathrm{WR}}(f) = \min\left\{r : f = \lim_{\varepsilon \to 0} f_\varepsilon \text{ for a sequence } f_\varepsilon \text{ with } \mathrm{WR}(f_\varepsilon) \leq r\right\}.$$

Clearly $\underline{\mathrm{WR}}(f) \leq \mathrm{WR}(f)$.
There are examples where the inequality is strict:

$$\mathrm{WR}(x^{d-1}y) = d$$
$$\underline{\mathrm{WR}}(x^{d-1}y) = 2.$$

Debordering border Waring rank

A *debordering result* for $\underline{\mathrm{WR}}$ is an inequality of the form

$$(\text{some complexity measure of } f) \le (\text{some function of } \underline{\mathrm{WR}}(f)).$$

Theorem. [Bläser-Dörfler-Ikenmeyer]

$$\mathrm{abpw}(f) \le \underline{\mathrm{WR}}(f).$$

Bold Conjecture.
For $f \in S^d\mathbb{C}^N$, then $\mathrm{WR}(f) \le O(d) \cdot \underline{\mathrm{WR}}(f)$.

- True for small $\underline{\mathrm{WR}}(f)$:
  [Sylvester, Segre, Buczynski-Landsberg, Ballico-Bernardi, Chiantini];
- True when $\underline{\mathrm{WR}}(f)$ nearly maximal:
  [Blekherman-Teitler].

Debordering border Waring rank - cont'd

Theorem. [DGIJL] For $f \in S^d\mathbb{C}^N$, if $\underline{\mathrm{WR}}(f) = r$, then

$$\mathrm{WR}(f) \leq d \cdot \binom{2r-2}{r-1}.$$

Idea of the proof:

Three ingredients:

(i) We may assume f can be written in r variables.

(ii) We may assume $\deg(f) \geq r$.

(iii) *Generalized additive decompositions* allow one to give bounds in this range.

Previously only general bounds were of the form $\mathrm{WR}(f) \leq O(d^r)$ or $\mathrm{WR}(f) \leq O(r^d)$ which is almost trivial using just "ingredient (i)".

Kumar's *product plus constant* model

Let $f \in \mathbb{C}[x_1, \dots, x_N]$. The *Kumar's complexity* of f is

$$\mathrm{Kc}(f) = \min\left\{ r : f = \alpha\left(\prod_1^r (1+\ell_j) - 1\right) \text{ for some } \ell_j \in S^1\mathbb{C}^N, \alpha \in \mathbb{C} \right\}$$

Example. Set $\omega = \exp(2\pi i/d)$.

$$\ell^d = (1+\omega^0\ell)\cdots(1+\omega^{d-1}\ell) - 1 \quad \text{so} \quad \mathrm{Kc}(\ell^d) = d.$$

However $\mathrm{Kc}(f)$ is not always finite. In fact, if f is homogeneous, then $\mathrm{Kc}(f)$ is finite if and only if $f = \ell^d$.

The *border Kumar's complexity* of f is

$$\underline{\mathrm{Kc}}(f) = \min\left\{ r : f = \lim_{\varepsilon \to 0} f_\varepsilon \text{ for a sequence } f_\varepsilon \text{ with } \mathrm{Kc}(f_\varepsilon) \le r \right\}$$

$\underline{\mathrm{Kc}}(f)$ is finite for every polynomial f.
Theorem. [Kumar]
For $f \in S^d\mathbb{C}^N$, one has $\underline{\mathrm{Kc}}(f) \le \deg(f) \cdot \mathrm{WR}(f)$.

A converse of Kumar's result

How good is the bound $\underline{\mathrm{Kc}}(f) \leq \deg(f) \cdot \mathrm{WR}(f)$?

Example.

$$x_1 \cdots x_n = \lim_{\varepsilon \to 0} \varepsilon^n \left(\prod_1^n (1 + \tfrac{1}{\varepsilon} x_j) - 1 \right)$$

One has

$$\underline{\mathrm{Kc}}(x_1 \cdots x_n) = n \qquad \mathrm{WR}(x_1 \cdots x_n) = 2^{n-1}.$$

Except for this case, $\underline{\mathrm{Kc}}$ is roughly equivalent to $\underline{\mathrm{WR}}$.

Theorem. [DGIJL]
For $f \in S^d \mathbb{C}^N$, either f is a product of linear forms or

$$\underline{\mathrm{WR}}(f) \;\leq\; \underline{\mathrm{Kc}}(f) \;\leq\; \deg(f) \cdot \underline{\mathrm{WR}}(f).$$

Generalizing the *product plus constant* model

For $i = 1, \ldots, r$, let X_i be an $m \times m$ matrix of linear forms. Then

$$A = (\mathrm{id}_m + X_1) \cdots (\mathrm{id}_m + X_r) - \mathrm{id}_m$$

is a matrix whose entries are (non-homogeneous) polynomials of degree d without constant term.

Idea: Fix m and define a complexity measure for f in terms of the value of r in the expression of A.

We recover the completeness of ABPs of width 3 for VF [Ben-Or and Cleve].

Theorem. [DGIJL] If $f \in S^d\mathbb{C}^N$ has a formula of depth δ, then f can be expressed as an entry of A for some $r \leq 4^\delta$ and $m = 3$.

Parity-alternating elementary symmetric functions

The d-th homogeneous component of A is

$$\overline{e}_d(X_1, \ldots, X_r),$$

the elementary symmetric polynomial in non-commuting variables.

Fix $m = 2$ and specialize $X_i = \left(\begin{smallmatrix} 0 & x_i \\ 0 & 0 \end{smallmatrix}\right)$ if i is odd, $X_i = \left(\begin{smallmatrix} 0 & 0 \\ x_i & 0 \end{smallmatrix}\right)$ if i is even. Let $C = \overline{e}_d(X_1, \ldots, X_r)$. One of the entries of C is

$$c_{r,d} = \textstyle\sum_{(i_1, \ldots, i_d)} x_{i_1} \cdots x_{i_d}$$

where the sum is over parity-alternating increasing sequences.

For $f \in S^d\mathbb{C}^N$, define

$$r_c(f) = \min\{r : f = c_{r,d}(\ell_1, \ldots, \ell_r) \text{ for some } \ell_i \in S^1\mathbb{C}^N\}$$

and let $\underline{r}_c$ be the corresponding *border* complexity.

Theorem. [DGIJL]
VNP $\not\subseteq \overline{\text{VQP}}$ if and only if $\underline{r}_c(\mathrm{perm}_m)$ grows super-quasipolynomially.

What next?

- Debordering Waring rank:
  - study the geometry of approximating curves;
  - explore other models equivalent to Waring rank.
- Homogeneous polynomials defining complexity classes:
  - GCT and obstructions;
  - geometric methods for orbit-closures.