Monotone Complexity of Spanning Tree Polynomial Re-visited

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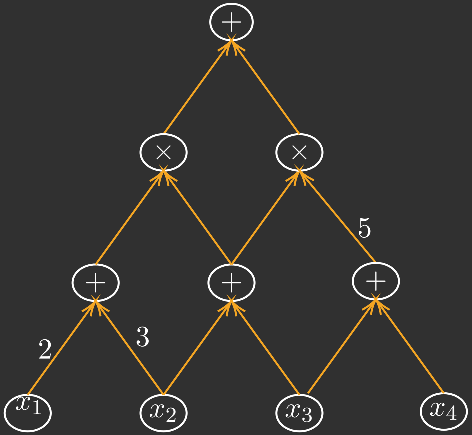
Summary

Basic Model of Computation

Arithmetic Circuits



- Arithmetic circuits are model for computing polynomials.
- Size of the circuit is the number of nodes.
- Monotone Circuits : Only non-negative scalars are allowed on edges. They naturally compute monotone polynomials.

$$f(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + 5x_3 + 5x_4)(x_2 + x_3)$$

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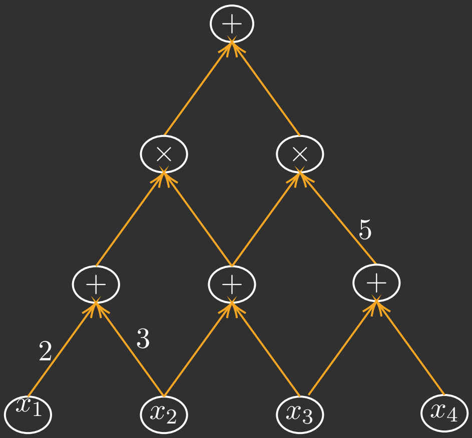

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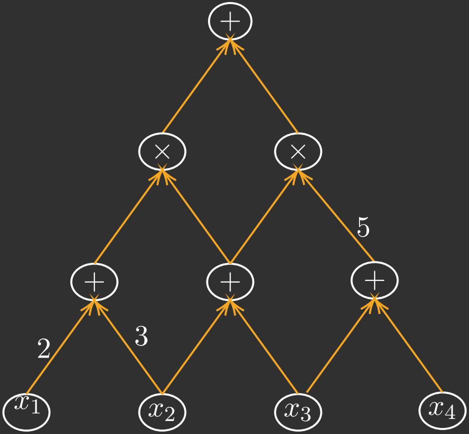

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Monotone Computation

Important monotone polynomials:

- $$S_{n,k} = \sum_{\substack{S \in [n] \\ |S|=k}} \prod_{i \in S} x_i$$
- $$\mathrm{Perm}_{n\times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}$$

Monotone circuits are universal for monotone polynomials.

$$f = \sum \alpha_m \, m \ , \ \alpha_m \geq 0$$

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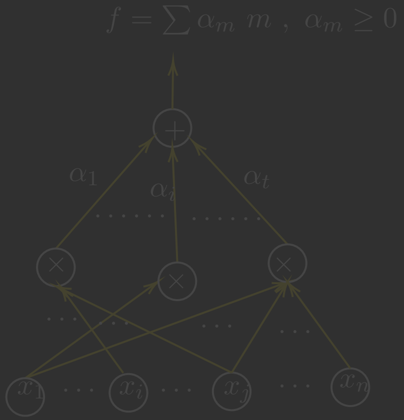

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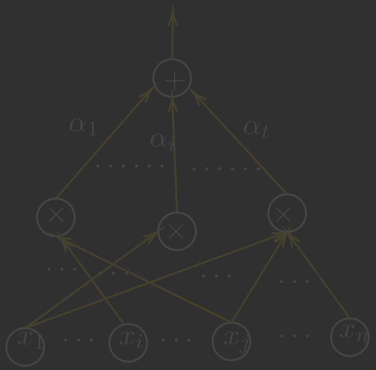

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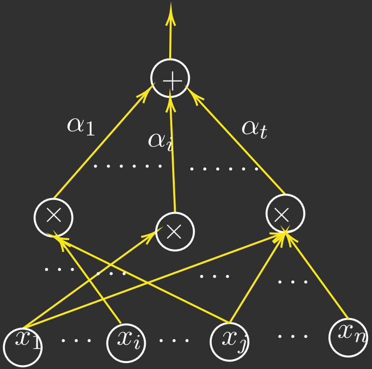

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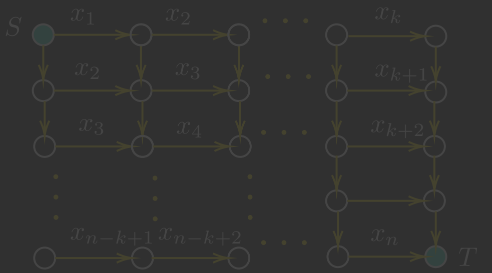

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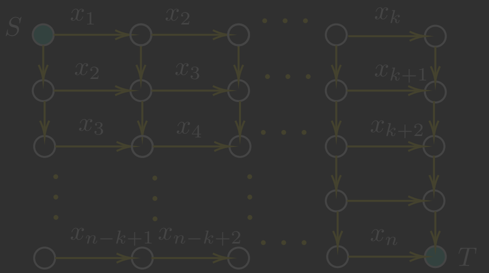

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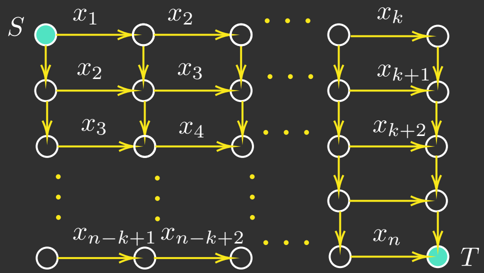

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- Number of variables in $\text{Perm}_{n\times n}$ is n^2.
- $\text{Ckt}^+ - \text{size}(\text{Perm}_{n\times n}) \geq 2^{\Omega(n)}$.
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Strongly Exponential Lower Bound Against Monotone Circuits

Known Results

Strongly exp. lower bound

- Gashkov-Sergeev (80's).
- Raz-Yehudayoff (2009).
- Srinivasan (2019)
- Cavalar-Kumar-Rossman (2020).
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All polynomials are in VNP.

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Yes!(Our result)

Our Result

Theorem:

The Spanning tree polynomial defined for a family of constant degree expander graphs on n vertices requires monotone circuits of size $2^{\Omega(n)}$.

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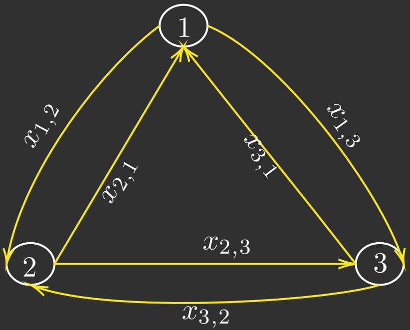
What is Spanning Tree Polynomial ?



- $\mathrm{ST}_3 = x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1}$.
- $x_{1,2} \cdot x_{3,2}$ is not a monomial in ST_3.

- $G = (V, E), |V| = n$ is bi-directed.
- T is the set of maps from $[2, \ldots, n]$ to $[n]$ that gives spanning tree rooted at 1.
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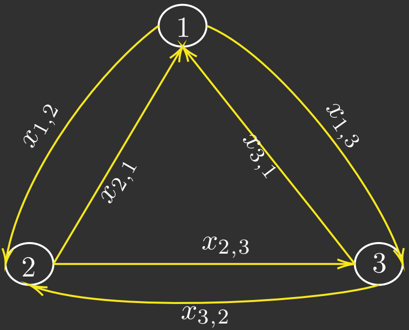

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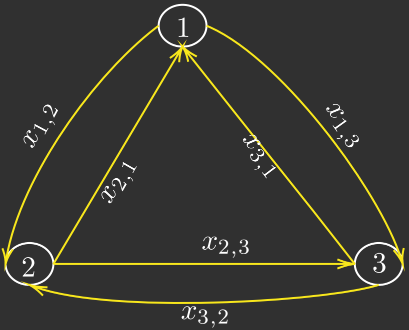

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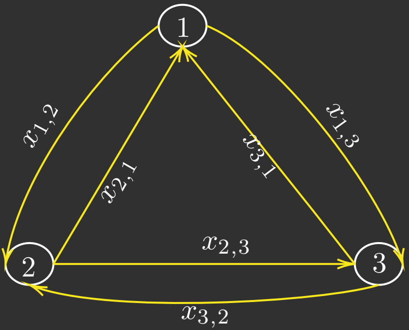

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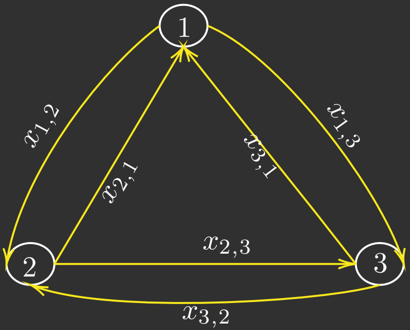

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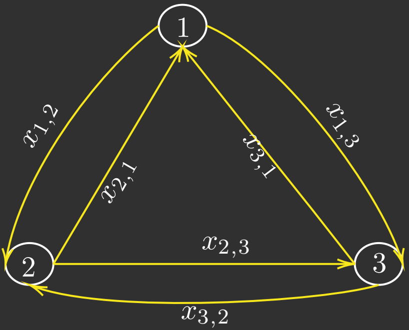

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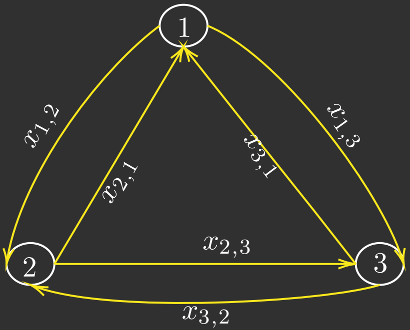

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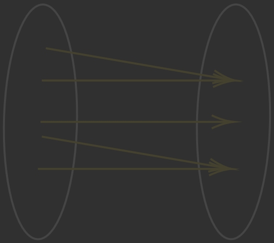
Set-multilinear Polynomial



$\pi : [2, n] \longrightarrow [n]$

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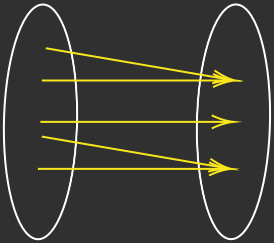

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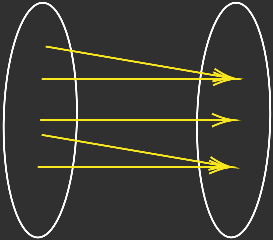

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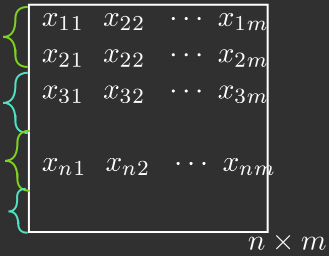
Set-multilinear Monotone Structure Theorem

For set-multilinear monotone polynomial f

if $C^{+}(f) = S$ then

$$f = \sum_{t=1}^{S+1} \alpha_t \cdot \beta_t$$



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$\forall\ t$ and

$|I(\alpha_t)|, |I(\beta_t)| \in \left[\frac{n}{3}, \frac{2n}{3}\right] \longleftarrow$ Nearly Balanced Partition

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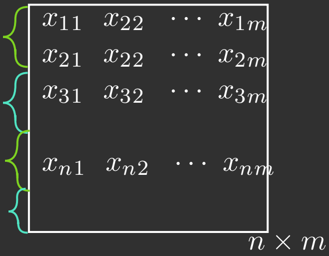

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Proof Idea of Result

- $\mathrm{ST}_n = \sum_{t=1}^{S+1} a_t \cdot b_t$.

- The measure is counting spanning tree monomials.

- Using Expander Mixing lemma on a d regular expander graph, $\exists\, C_1$ s.t. $|\mathrm{mon}(a_t \cdot b_t)| \leq (C_1 d)^{n-1}$ for any t.

- The non spanning tree monomials are forbidden.

- Using Matrix Tree theorem $\exists\, C_2$ s.t. $|\mathrm{mon}(\mathrm{ST}_n)| \geq (C_2 d)^{n-1}$.

- $C_2 > 2C_1 \implies S \geq 2^{\Omega(n)}$.

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- The measure is counting spanning tree monomials.
- The non spanning tree monomials are forbidden.

- $\mathrm{ST}_n = \sum_{t=1}^{S+1} a_t \cdot b_t$.
- Using Expander Mixing lemma on a d regular expander graph, $\exists\ C_1$ s.t. $|\mathrm{mon}(a_t \cdot b_t)| \leq (C_1 d)^{n-1}$ for any t.
- Using Matrix Tree theorem $\exists\ C_2$ s.t. $|\mathrm{mon}(\mathrm{ST}_n)| \geq (C_2 d)^{n-1}$.
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ϵ-Sensitive Monotone Lower Bound

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Problem

Can monotone l.b yield general circuit lower bound ?

Remark

Boolean world : Slice function (Valiant 1986)

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Is there a hard polynomial f s.t. for every $\epsilon > 0$, the polynomial $g_\epsilon = E + \epsilon \cdot f$ has large monotone complexity ? $E \longrightarrow$ *Easy for monotone.*

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- If $\epsilon = 0$ then g_ϵ has trivial monotone circuit.
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Results on ϵ-Sensitive Monotone Lower Bounds

- C.D.M (2021): First ϵ-sensitive monotone l.b against a VNP polynomial family $\{f_n\}$ with $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

- Can we show this for VP polynomial ?

Theorem:
The Spanning Tree polynomials for complete graph on n vertices require exponential size ϵ-sensitive monotone lower bound in the set-multilinear setting for $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

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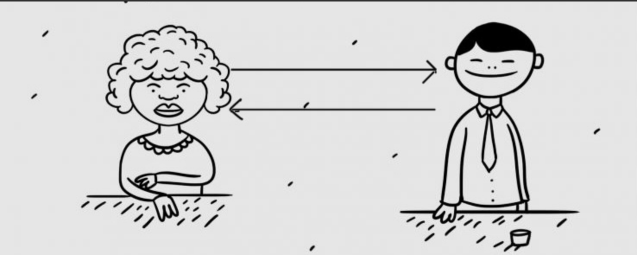
Spanning Tree Communication Problem



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$E_A = \{(u \to v) \mid u \in V_A\}$ $\quad \{(w \to v) \mid w \in V_B\} = E_B$

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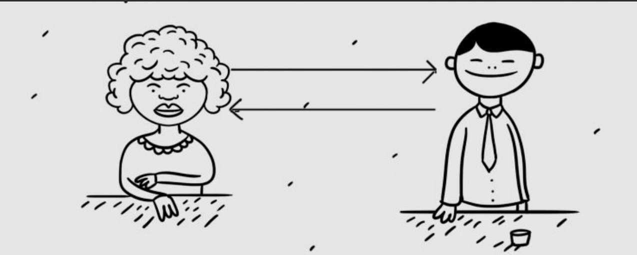

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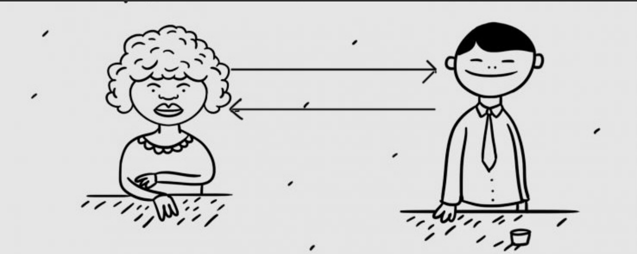

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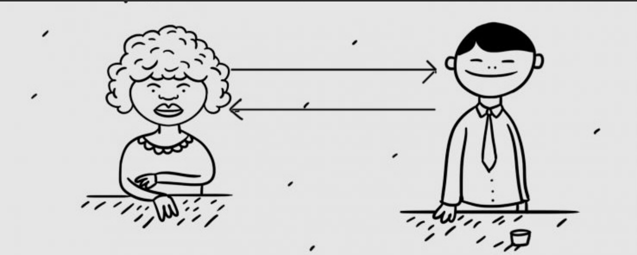

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Spanning Tree is Hard Under a Fixed Partition

- A gadget reduction from the Inner Product problem to the Spanning Tree problem.
- Inner Product: $\mathsf{IP}(x, y) = \sum_{i=1}^{n} x_i \cdot y_i \pmod 2$ is a well known hard problem.
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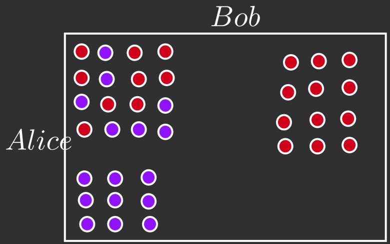

$R = A \times B$ where $A, B \subseteq \{0,1\}^m$

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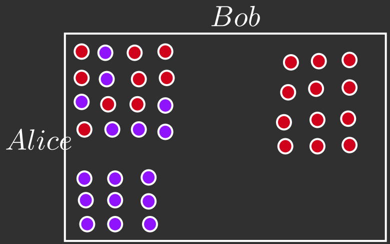

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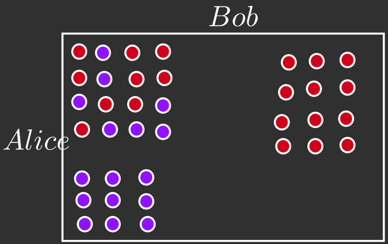

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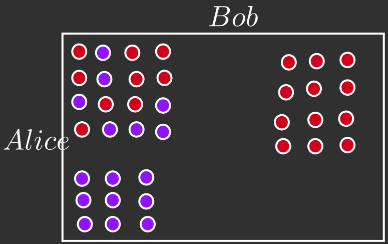

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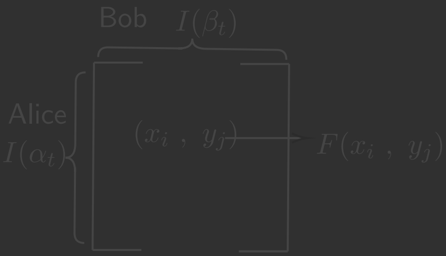

$(I(\alpha_t), I(\beta_t)) = [n]$

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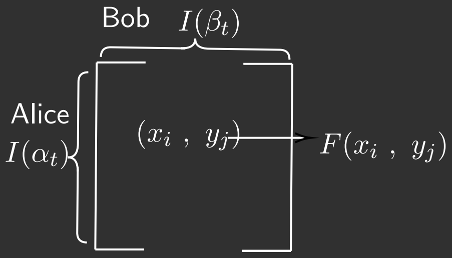

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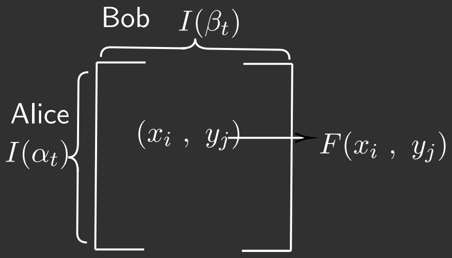

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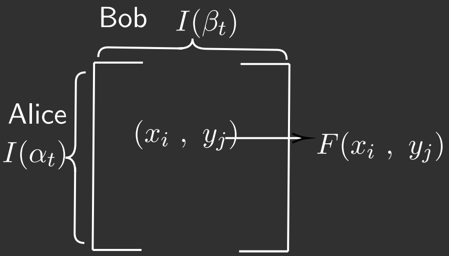

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There are more exciting open problems in our paper. We invite you to check the following link
https://arxiv.org/abs/2109.06941

Thank You