Degree-Restricted Strength Decompositions and Algebraic Branching Programs

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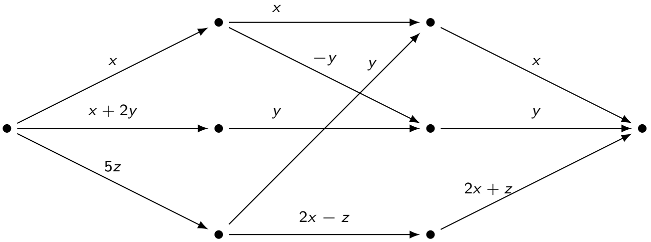
Homogeneous algebraic branching programs

We consider homogeneous algebraic branching programs with linear forms on edges



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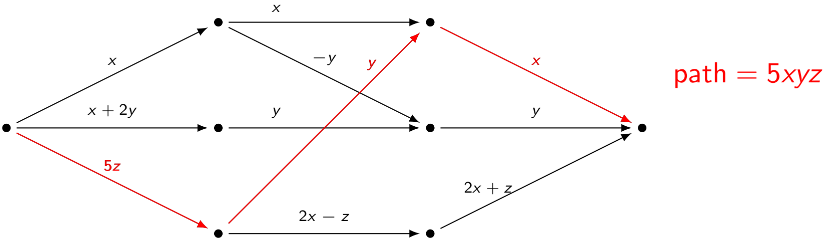
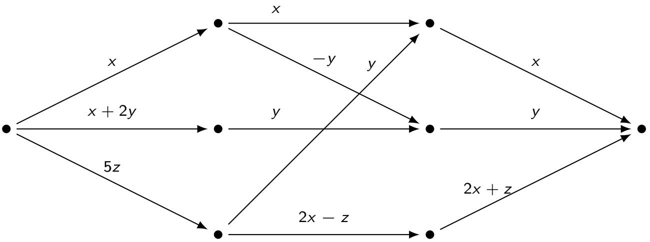


- Weight of a path is the product of all labels on this path
- The polynomial computed by an ABP is the sum of weights over all paths from the source to the sink
- ABP size is the number of internal vertices

Algebraic branching programs

- Computational power of algebraic branching programs is intermediate between formulas and circuits
- The model is very convenient algebraically because ABPs are connected to iterated matrix multiplication
- Concatenation of paths is the same as matrix multiplication



$$\begin{bmatrix} x & x+2y & z \end{bmatrix} \times \begin{bmatrix} x & -y & 0 \\ 0 & y & 0 \\ y & 0 & 2x-z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 2x+z \end{bmatrix}$$

Algebraic branching programs in noncommutative setting

- Algebraic branching programs were first formally introduced by Nisan in the noncommutative setting
- Noncommutative ABPs are very rigid and have a very nice algebraic characterization
- Nisan computes the noncommutative ABP complexity exactly in terms of ranks of partial derivative matrices

N. Nisan. *Lower Bounds for Non-Commutative Computation: Extended Abstract.* STOC'91

Algebraic branching programs in commutative setting

- Commutative algebraic branching programs are more complicated
- A quadratic lower bound for homogeneous ABPs was proven by Kumar

Theorem (Kumar)

$$\text{homABP-size}(x_1^d + x_2^d + \cdots + x_n^d) \geq (d-1)\lceil \tfrac{n}{2} \rceil$$

- Chatterjee, She, Kumar and Volk extend this bound to non-homogeneous ABPs

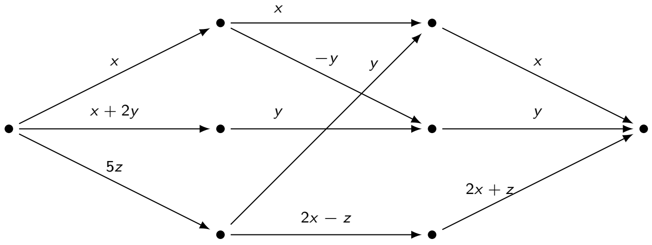
M. Kumar. *A Quadratic Lower Bound for Homogeneous Algebraic Branching Programs.* CCC 2017 / comput. complex. 28(3)
P. Chatterjee, M. Kumar, A. She, and B. L. Volk. *A Quadratic Lower Bound for Algebraic Branching Programs.* CCC 2020

Algebraic branching programs and decompositions

- A path from the source to the sink contains a vertex in layer k
- This gives a decomposition of the form

$$F = \sum_{j=1}^{r} G_j H_j$$

- G_j, H_j are homogeneous, $\deg G_j = k$, $\deg H_j = d - k$
- Number of summands is equal to the number of vertices in layer k



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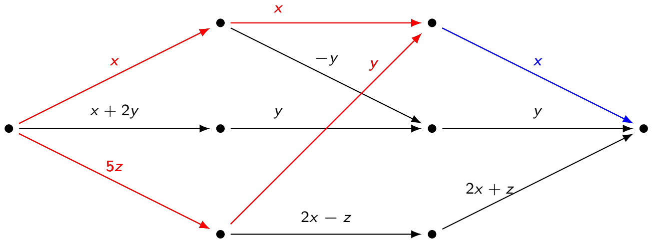

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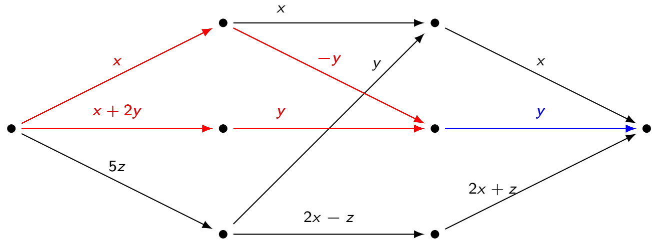

$$F = (x^2 + 5yz) \cdot x + 2y^2 \cdot y$$

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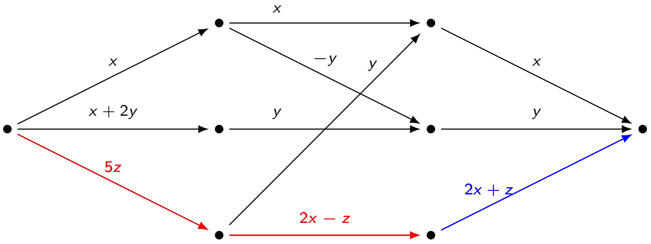

$$F = (x^2 + 5yz) \cdot x + 2y^2 \cdot y + (10xz - 5z^2) \cdot (2x + z)$$

Rank decompositions in noncommutative case

- In the noncommutative case: write

$$F = \sum f_{\mathbf{ab}} \left(x_{a_1} \otimes \cdots \otimes x_{a_k}\right) \otimes \left(x_{b_1} \otimes \cdots \otimes x_{b_{d-k}}\right)$$

- Construct a matrix $F_k = (f_{\mathbf{ab}})$
- $F = \sum_{j=1}^{r} G_j \otimes H_j$ correspond to rank decompositions of this matrix
- This proves a lower bound part of Nisan's result

Strength decompositions

- In the commutative case:
- Decompositions $F = \sum_{j=1}^{r} G_j H_j$ with homogeneous G_j and H_j were studied before in algebra and algebraic geometry
- The minimal number of summands in such a decomposition is called the *strength* $\mathrm{str}(F)$

Theorem

$\text{homABP-size}(F) \geq (d-1) \cdot \mathrm{str}(F)$

- We define *k-restricted strength* $\mathrm{str}_k(F)$ as the minimal number of summands in a strength decomposition with $\deg G_j = k$

Theorem

$\text{homABP-size}(F) \geq \sum_{k=1}^{d-1} \mathrm{str}_k(F)$

- A special case is the *slice rank* of a polynomial $\mathrm{str}_1(F)$

Strength and singular locus

Definition

The *singular locus* $\mathrm{Sing}(F)$ is the variety defined by equations $\frac{\partial F}{\partial x_i} = 0$.

- If $F = \sum_{j=1}^{r} G_j H_j$, then $\frac{\partial F}{\partial x_i} = \sum_{j=1}^{r} \left[G_j \frac{\partial H_j}{\partial x_i} + H_j \frac{\partial G_j}{\partial x_i} \right]$
- If all $G_j = H_j = 0$, then all $\frac{\partial F}{\partial x_i} = 0$
- $2r \geq \operatorname{codim} \mathrm{Sing}(F)$

Theorem

$\mathrm{str}(F) \geq \frac{1}{2} \operatorname{codim} \mathrm{Sing}(F)$

- This is the essence of Kumar's lower bound
- This cannot give bounds better than $\mathrm{str}(F) \geq \lceil \frac{N}{2} \rceil$, where N is the number of variables
- Can we do better?

Degree-restricted strength and subvarieties

- Suppose $F = \sum_{j=1}^{r} G_j H_j$.
- Let Z be the hypersurface defined by $F = 0$.
- Z contains the variety $X = \{G_1 = \cdots = G_r = 0\}$
- If $\deg G_j = k$, then the degree of X is at most k^r (essentially by Bezout theorem)

Theorem

If Z does not contain linear subspaces of codimension c, then

$$\mathrm{str}_1(F) \geq c + 1$$

If Z does not contain subvarieties of codimension c and degree $< s$, then

$$\mathrm{str}_k(F) \geq \min\{c + 1, \log_k s\}$$

Result for explicit polynomials

- Consider the polynomials

$$P_{n,d} = x_0^d + x_1 x_2^{d-1} + x_3 x_4^{d-1} + \cdots + x_{2n-1} x_{2n}^{d-1}$$

- The number of variables is $N = 2n + 1$
- Singular locus lower bound gives $\mathrm{str}(P_{n,d}) \geq \lceil \frac{n+1}{2} \rceil \approx \frac{N}{4}$

Theorem

$$\mathrm{str}_1(P_{n,d}) = n + 1 \approx \frac{N}{2}$$

$$\mathrm{str}_k(P_{n,d}) \geq \min\{n + 1, \log_k d\}$$

- This improves Kumar's lower bound $\approx (d+1)\frac{N}{4}$ by additive term

$$\approx \frac{N}{2} + \frac{N}{2} \cdot d^{\mathrm{const}/N}$$

Intersection theory

- We use intersection theory to prove the result on $P_{n,d}$
- For a variety Z, the *Chow group* $\mathrm{CH}_a(Z)$ consists of formal linear combinations of dimension a irreducible subvarieties modulo an equivalence relation called *rational equivalence*
- Rational equivalence can be though of as an existence of a certain kind of deformation from one collection of varieties to another

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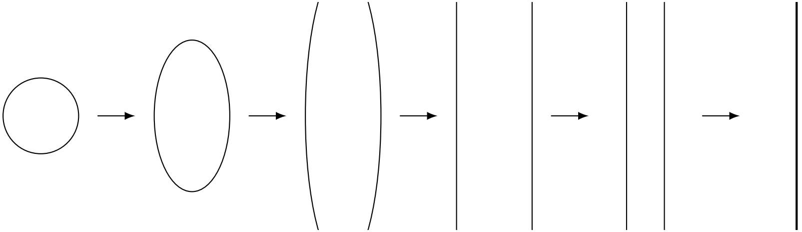

$$[\mathit{conic}] = 2[\mathit{line}] \text{ in } \mathrm{CH}_1(\mathbb{P}^2)$$

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Lemma

There is a homomorphism $\mathrm{CH}_a(P_{n,d}) \to \mathrm{CH}_{a+1}(P_{n+1,d})$, *which preserves the degree*

Shioda polynomials and slice rank

- Consider the polynomials

$$S_{n,d} = x_0x_1^{d-1} + x_1x_2^{d-1} + \cdots + x_{n-1}x_n^{d-1} + x_nx_0^{d-1} + x_{n+1}^d$$

- The number of variables is $N = n + 2$
- We call them *Shioda polynomials* because they were studied by Shioda in intersection theory
- The singular locus lower bound is $\mathrm{str}(S_{n,d}) \geq \lceil \frac{N}{2} \rceil$
- For $S_{4,d}$ we have $\mathrm{str}(S_{4,d}) \geq 3$

Theorem

$\mathrm{str}_1(S_{4,d}) = 4$

- This is the first lower bound better than $\lceil \frac{N}{2} \rceil$ for an explicit polynomial
- This improves the Kumar's lower bound on hom. ABP size by $+2$

Open questions

- Is analysis of subvarieties useful for other complexity questions?
- Can we determine the exact complexity of $P_{n,d}$ and $S_{n,d}$?
- Can we at least prove $\mathrm{str}_1(S_{n,d}) = \frac{n+1}{2} + 1$ for all even n?
- Is computing strength NP-hard?
- Does existence of explicit polynomials with high strength have complexity implications?