Demystifying the border of depth-3 circuits

Joint works with Pranjal Dutta & Prateek Dwivedi. [CCC'21, FOCS'21, FOCS'22]

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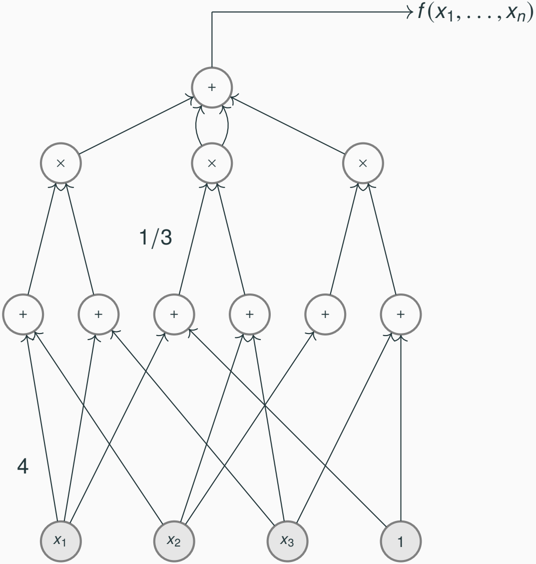
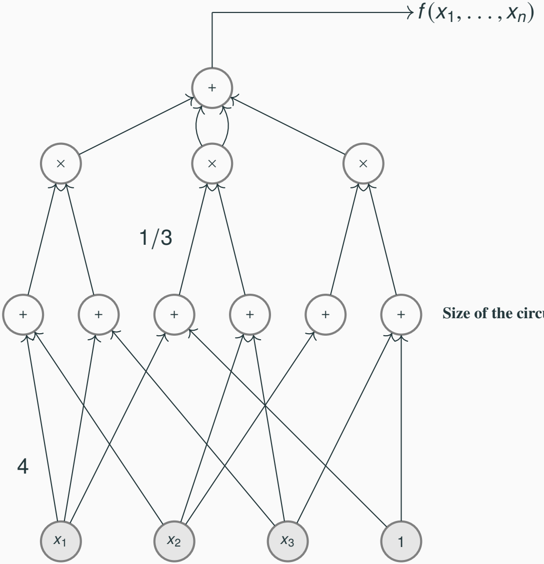
Algebraic circuits– VP

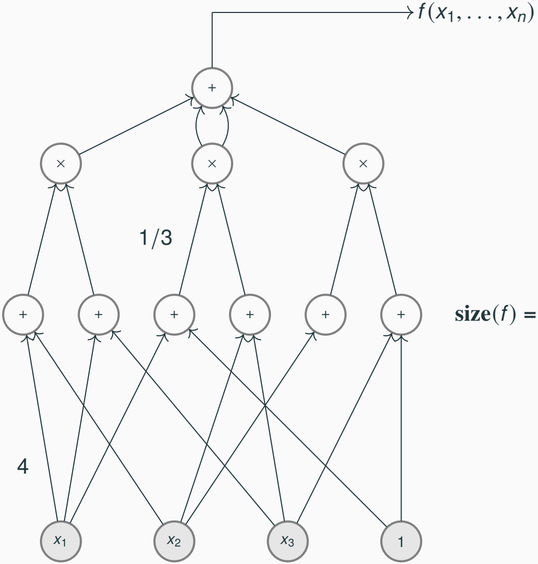


Algebraic circuits– VP



Size of the circuit = number of nodes + edges

Algebraic circuits– VP



size(f) = min size of the circuit computing f

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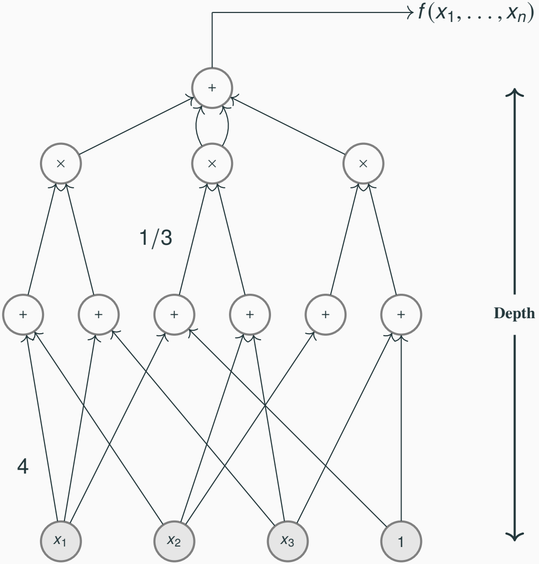

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- VBP: The class VBP is defined as the set of all sequences of polynomials $(f_n)_n$ with polynomially bounded $\mathrm{dc}(f_n)$.

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- Are there hard polynomial families $(f_n)_n$ such that it cannot be computed by an n^c-size circuit, for *every* constant c? i.e. $\text{size}(f_n) = n^{\omega(1)}$?

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- The minimum dimension of the matrix to compute f, is called the **permanental complexity** $\text{pc}(f)$.

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VNP = "hard to compute?" [Valiant 1979]

The class VNP is defined as the set of all sequences of polynomials $(f_n(x_1, \ldots, x_n))_{n \geq 1}$ such that $\mathsf{pc}(f_n)$is bounded by n^c for some constant c.

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$\mathrm{VBP} \neq \mathrm{VNP}$ & $\mathrm{VP} \neq \mathrm{VNP}$. Equivalently, $\mathrm{dc}(\mathrm{perm}_n)$ and size(perm_n) are both $n^{\omega(1)}$.

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  - Assuming GRH (Generalized Riemann hypothesis), the results hold over $\mathbb{C}$ as well.

Border Complexity and GCT

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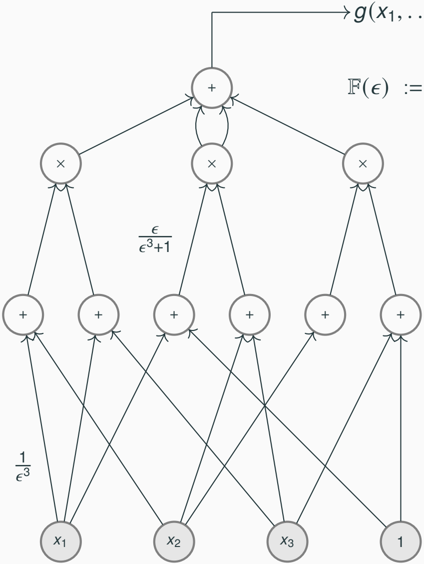
- This motivates a new model: *'approximative circuit'*.

Approximative circuits



$g(x_1, \ldots, x_n, \epsilon) \in \mathbb{F}(\epsilon)[\boldsymbol{x}]$

$\mathbb{F}(\epsilon) := \left\{\frac{p(\epsilon)}{q(\epsilon)} \mid p, q \in \mathbb{F}[\epsilon], q(\epsilon) \neq 0\right\}$ is the fn.field

Algebraic approximation

- Suppose, we assume the following:
  - $g(\boldsymbol{x}, \epsilon) \in \mathbb{F}[x_1, \ldots, x_n, \epsilon]$, i.e. it is a polynomial of the form

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- Summary: g_0 is **non-trivially** 'approximated' by the circuit, since $\lim_{\epsilon \to 0} g(\boldsymbol{x}, \epsilon) = g_0$.

Algebraic approximation— $\overline{\mathrm{VP}}$

Algebraic Approximation [Bürgisser 2004]

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- Curious eg.: Complexity of degree s factor of a size-s polynomial?

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  - What about border depth-3 circuits (both upper bound and lower bound)?

Border Depth-3 Circuits

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- How powerful are $\Sigma^{[k]}\Pi\Sigma$ circuits, for constant k? Are they *universal*?

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- They compute polynomials (*not necessarily* homogeneous) of the form $\sum_{i=1}^{k}\prod_{j=1}^{d_i} \ell_{ij}$, where ℓ_{ij} are linear polynomials (i.e. $a_0 + a_1x_1 + \ldots + a_nx_n$, for $a_i \in \mathbb{F}$).
- Product fan-in $= \max d_i$.
- How powerful are $\Sigma^{[k]}\Pi\Sigma$ circuits, for constant k? Are they *universal*?
- Impossibility result: The *Inner Product* polynomial $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + \ldots + x_{k+1}y_{k+1}$ **cannot** be written as a $\Sigma^{[k]}\Pi\Sigma$ circuit, *regardless* of the product fan-in (even allowing $\exp(n)$ product fan-in!).

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- How about $\overline{\Sigma^{[k]}\Pi\Sigma}$?

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$$P + \epsilon^d \cdot R(\boldsymbol{x}, \epsilon) = -\epsilon^{-d} + \epsilon^{-d} \cdot \prod_{i=1}^{m} \prod_{j=1}^{d} (\alpha_j + \epsilon \cdot \ell_i) \in \Sigma^{[2]}\Pi^{[md]}\Sigma .$$

Proving Upper Bounds

De-bordering $\overline{\Sigma^{[2]}\Pi\Sigma}$ circuits

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- We devise a technique called DiDIL - **Di**vide, **D**erive, **I**nduct with **L**imit.

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- Both $\Phi(T_1)$ and $\tilde{T}_2$ have $\Pi\Sigma$ circuits (they have z and ϵ).

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- $\deg(f) = d \implies \deg_z(\Phi(f)) = d \implies \deg_z(\partial_z(\Phi(f))) = d - 1$.
- Suffices to compute $g_1 \bmod z^d$ and take the limit!

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$$\begin{aligned} \lim_{\epsilon \to 0} g_1 \mod z^d &\equiv \lim_{\epsilon \to 0} \Pi\Sigma/\Pi\Sigma \cdot \left(\sum \mathrm{dlog}(\Sigma)\right) \mod z^d \\ &\equiv \lim_{\epsilon \to 0} (\Pi\Sigma/\Pi\Sigma) \cdot (\Sigma \wedge \Sigma) \mod z^d \\ &\in \overline{(\Pi\Sigma/\Pi\Sigma) \cdot (\Sigma \wedge \Sigma)} \mod z^d\,. \end{aligned}$$

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- Eliminate division, and integrate (interpolate wrt z) to get $\Phi(f)/t_2 = \mathsf{ABP} \implies \Phi(f) = \mathsf{ABP} \implies f = \mathsf{ABP}$.

Proving Lower Bounds

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  - Since, $\mathsf{IMM}_{n,d} \in \mathsf{VBP}$, $\overline{\Sigma^{[k]}\Pi\Sigma} \neq \mathsf{VBP}$.

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- What does work (if at all!)?

Our results

[Dutta-Saxena FOCS'22]

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- Classical is about *impossibility*. While, border is about *optimality*.

Non-homogeneity is 'bad'

- Recall the non-border lower bound proof, of making an ideal $\mathcal{I}_k = \langle \ell_1, \ldots, \ell_k \rangle$, such that $f \neq 0 \bmod \mathcal{I}_k$, but $\Sigma^{[k]}\Pi\Sigma = 0 \bmod \mathcal{I}_k$.

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- Lesson: Taking mod blindly fails *miserably*!
- The worst case:

$$f + \epsilon S \;=\; T_1 + T_2 \,,$$

where T_i has each linear factor of the form $1 + \epsilon\ell$!

Non-homogeneity is all we need to care

- Three cases to consider:
  - Case I: Each T_1 and T_2 has one linear polynomial $\ell_i \in \mathbb{F}(\epsilon)[\boldsymbol{x}]$ as a factor, whose ϵ-free term is a linear form. Example: $\ell = (1+\epsilon)x_1 + \epsilon x_2$,

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- So, all-non-homogeneous is all we have to handle!

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- Partial-derivative measure shows that the above implies $s \geq 2^{\Omega(d)}$!

Conclusion

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Thank you! Questions?