Towards refining the No Occurence Obstructions in GCT

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Table of Contents

The GCT approach

Orbit closure membership

- Let $W = \mathbb{C}^n$, and let $\mathbb{P}(S^d(W^*))$ denote the projective space of homogeneous polynomials of degree d over W.
- $GL(W) \circlearrowright \mathbb{P}(S^d(W^*))$, a natural action

$$(f, \mathcal{M}) \to f \circ \mathcal{M}^T$$

- $\Omega_f := f \circ GL(W) \subseteq \mathbb{P}(S^d(W^*))$, the orbit of f, $\overline{\Omega_f}$ its Zariski closure

Fundamental problem of algebraic complexity

Given $f, g \in \mathbb{P}(S^d(W^*))$ is $g \in \overline{\Omega_f}$?

This problem is related to the *P vs NP* problem in complexity theory.

Actions on polynomials

$G = GL(2)$, V, polynomials of degree 2 in $\{x_1, x_2\}$.

- ❑ $g = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, $f_1 = x_1^2$.
    - $g \cdot f_1 = (x_1 + 2x_2)^2 = x_1^2 + 4x_2^2 + 4x_1x_2$.
- ❑ $g = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $f_2 = x_1x_2$.
    - $g \cdot f_2 = (x_1 + 2x_2)(3x_1 + x_2) = 3x_1^2 + 7x_1x_2 + 2x_2^2$

Determinant versus Permanent

- $W = \mathbb{C}^{n^2}$, $f = Determinant(x_{11}, \ldots, x_{nn}) \in S^n(x_{11}, \ldots, x_{nn})$
  - The stabilizer of Determinant is $S(GL(n) \times GL(n)) \rtimes \mathbb{Z}_2 \subseteq SL(n^2)$, (A, B) sending X to AXB, $\mathbb{Z}_2$ sending X to X^T.
  - The stabilizer of Determinant, G_{Det}, is reductive.

- $W = \mathbb{C}^{m^2}$, $f = Permanent(x_{11}, \ldots, x_{mm}) \in S^m(x_{11}, \ldots, x_{mm})$
  - The stabilizer of Permanent: $(M_n, M_n) \rtimes \mathbb{Z}_2 \subseteq SL(m^2)$, M_n being monomial matrices.
  - The stabilizer of Permanent, G_{Perm}, is reductive.

- The holy grail of algebraic complexity Let $m < n$. Is $x_{nn}^{n-m} Perm_m \in \overline{Det_n}$?

- Conjecture: [Valiant 79, Mumuley-Sohoni 02] Not true when n is subexponential in m

Reductive stabilizers

- Is $x_{nn}^{n-m} Perm_m \in \overline{O(Det_n)}$?

- The GCT approach - rests on the fact that the forms $(Det_n, Perm_m)$ have distinctive reductive stabilizers, which characterize the form - any polynomial with the same stabilizer as Det_n is a multiple of Det_n.

- G_{Det} reductive implies the orbit $GL(W)/G_{Det}$ is an affine variety, [Matsushima].

- The coordinate ring of the orbit of Determinant is $\mathbb{C}[W]^{G_{Det}}$

    - The boundary of the closure of an affine variety is empty or has pure codimension one.

- The symmetries of $Det_n, Perm_m$, should help us solve Valiant's conjecture.

$$\mathbb{C}[\overline{O(Det_n)}] \to \mathbb{C}[\overline{O(x_{nn}^{n-m} perm_m)}] \to 0$$

Information about $x_{nn}^{n-m} Perm_m$ not being in the orbit closure of Det_n should be present in their coordinate rings

Representation theoretic obstructions

- The $SL(n^2)$ orbit of Det_n is closed, we say it is stable.
- The $SL(m^2)$ orbit of $Perm_m$ is closed. $Perm_m$ is stable. $x_{nn}^{n-m} Perm_m$ is NOT stable
- Each homogeneous piece of their coordinate rings is a representation of $GL(W)$.
- $GL(W) \to GL(\mathbb{C}[\overline{O(Det_n)}]_d)$, a group homomorphism.
- $GL(W)$-representations are characterized by combinatorial data-like how an integer splits into its prime factors. The prime representations are called irreducible representations. The number of times one such irreducible representation occurs is its multiplicity.
- Multiplicities of representations as obstructions

If the multiplicity of an irreducible $GL(W)$ module V_λ occurring in $\mathbb{C}[\overline{O(x_{nn}^{n-m} Perm_m)}]_d$ is more than the multiplicity of V_λ in $\mathbb{C}[\overline{O(Det_n)}]_d$, $x_{nn}^{n-m} Perm_m$ is not in the orbit closure of det_n [Mulmuley-Sohoni]

- No Occurrence Obstruction Conjecture: When n is subexponential in m, for infinitely many d, there are irreducible representations which occur in $\mathbb{C}[\overline{O(x_{nn}^{n-m} Perm_m)}]_d$ but do not occur in $\mathbb{C}[\overline{O(Det_n)}]_d$.

No Occurrence obstruction

- [Ikenmeyer, Panova,17]
- [Bürgisser, Ikenmeyer, Panova,18]
- When $n > m^{26}$, every irreducible representation occurring in $\mathbb{C}[\overline{O(x_{nn}^{n-m} perm_m)}]_d$ occurs in $\mathbb{C}[\overline{O(det_n)}]_d$.

No occurrence as stated is not true.

[Adsul, Sohoni, S,22] A geometric approach to arrive at obstructions.

- Examines the limiting process of $y \to z$
- In the neighbourhood of z a *local model* with an explicit $\mathcal{G}$-action.
- As a consequence a Lie theoretic version of Luna's slice theorem, which works even when stabilizer H of z is not reductive.
- Analyses how the Lie algebra $\mathcal{K}$ of the stabilizer K of y and the Lie algebra $\mathcal{H}$ of H interact.

Our results: Joint with Adsul, Sohoni

- A conceptual proof of why no occurrence obstruction is not true.

What is needed to refine this?

- A better understanding of the limiting process $\mathcal{K} \to \mathcal{H}$.
- When z is in the G-closure of y, a more nuanced understanding of

$$0 \to \frac{I_z}{I_y} \to \mathbb{C}[\overline{O(y)}] \to \mathbb{C}[\overline{O(z)}] \to 0$$

- How to analyze the kernel $\frac{I_z}{I_y}$?
- Intermediate G-stable varieties could help to do better book keeping.
- Are there natural G-stable intermediate varieties?
- Two such constructions when z is the limit of y under a 1-PS λ.

$W(\lambda)$, which gives a thickening of $O(z)$ in the direction λ and allows a filtration of the kernel of A_y/A_z.

$Z_d(\lambda)$ which contains all limits z' which can be obtained a from a point in the orbit of y as a leading term of degree d.

No occurrence - a simpler proof

- There are forms in the orbit closure of the determinant which are stable under a large subgroup of $GL(W)$ and have trivial stabilizers.

Definition

A 1-PS of $GL(W)$ is a homomorphism of groups $\mathbb{C}^* \to GL(W)$.

Action of a 1-PS on forms

$$\lambda : t \to \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \qquad \mu : t \to \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

$$\lambda(t) \cdot (x^2 + y^2) = t^2x^2 + t^2y^2,$$
$$\mu(t) \cdot (x^2 + y^2) = t^2x^2 + t^{-2}y^2$$

- $\lambda(t)$ drives $(x^2 + y^2)$ to zero in $Sym^2(W^*)$.
- In $\mathbb{P}Sym^2(W^*)$, $\lambda(t)$ fixes $x^2 + y^2$.
- In $\mathbb{P}Sym^2(W^*)$, via $\mu(t)$ both x^2 and y^2 are picked up in the orbit closure of $x^2 + y^2$. These forms are leading terms of a 1-PS acting on $x^2 + y^2$

Proof sketch

- Hilbert-Mumford-Kempf criterion f is unstable if is a 1-PS $\lambda : \mathbb{C}^* \to SL(W)$ driving f to zero - there exists λ, 1-PS with leading term $\hat{f}$ of weight > 0. Semistable otherwise. Stable if in addition the orbit is closed - there are both positive and negative weights under every 1-PS.

Lemma

Let $B(Y) \in Sym^d(\mathbb{C}Y)$ and $B' \in Sym^e(\mathbb{C}Y)$ be two forms which are both stable, i.e., their $SL(Y)$-orbits are closed. Then the $SL(Y)$-orbit of the product $B \cdot B' \in Sym^{d+e}(\mathbb{C}Y)$ is also closed.

Proof:

- Otherwise, by the Hilbert-Mumford-Kempf theory, there exists $\lambda(t) \in SL(Y)$, with $wt(\widehat{BB'}) \geq 0$.
- But B and B' are stable. So $wt(\hat{B}), wt(\hat{B'}) < 0$. Since $\widehat{BB'} = \hat{B}\hat{B'}$ we must have $wt(\widehat{BB'}) = wt(\hat{B}) + wt(\hat{B'}) < 0$. □

Proof sketch – continued

- $n = 2m$, $X = \{X_{ij} | 1 \le i, j \le n\}$, $Y = \{X_{ij} | 1 \le i, j \le m\}$,X=($x_{ij}$)
- $B = det(Y)$. Let $A \in GL(\mathbb{C}Y)$ and let $B' = det(AY)$.
- BB' is stable within $Sym^a(\mathbb{C}Y) \subset V = Sym^a(\mathbb{C}X)$.
- Let X' be

$$\begin{bmatrix} Y & 0 \\ 0 & AY \end{bmatrix}$$

- $det(X') = BB'$.
- There exists $g \in GL(\mathbb{C}X)$ and a 1-PS $\mu(t) \in GL(\mathbb{C}X)$ such that $\widehat{gdet(X)}$ under μ is BB'.
- $G_{BB'} = G_{det_m} \cap G_{det(AY)} = G_{det_m} \cap (A^{-1} G_{det_m} A)$.
- There exists A for which the above is trivial, only identity element. - example $A = diag(t^{2^i}), 1 \le i \le a^2$ - generic matrix in $GL(\mathbb{C}X)$

there is a $SL(\mathbb{C}Y)$-stable form with trivial stabilizer in the orbit closure of Det_{2m}.

Theorem

Let $V_\lambda(\mathbb{C}^{m^2})$ be an irreducible Weyl module with rows not exceeded m^2, then $V_\lambda(\mathbb{C}^{(2m)^2}))$ is present in $\mathbb{C}[\overline{O_V(det(X))}]$.

Sketch.

- The algebraic Peter Weyl Theorem, tells us that every $V_\lambda(\mathbb{C}^{m^2})$ with at most m^2-parts occurs in the coordinate ring of the orbit of BB', since its stabilizer is trivial.
- Every such module occurs in the $GL(m^2)$ orbit of BB' since BB' is stable for $SL(\mathbb{C}Y)$, [MS 01][BMLW, 12].
- Every such module now occurs in the $GL((2m)^2)$-orbit closure of BB', (Lifting Lemma)
- So each such module occurs in $GL((2m)^2)$-orbit closure of $Det(X)$

□

Stabilizer limits

Assumptions

- V is a $GL(X)$-representation with $tId \cdot v = t^c v$.
- y is a stable form.
- $\lambda(t)y = t^d y_d + t^e y_e + higher\ terms$. $z := \hat{y} = y_d$ is the leading term picked up in the projective orbit closure by a 1-PS.
- y_e is not in the orbit of z.
- K is the stabilizer of y and H that of z.
- Note that if a form z is an affine projection of $y := Det_n$, then there is a 1-PS acting λ such that $\hat{y} = z$. Studying limits picked up by 1-PS is relevant and useful.

Lie algebras - a quick recap

- $G := GL(\mathbb{C}X)$ is a Lie group, it has the structure of a complex manifold. The tangent space at Id is $\mathcal{G} := End(\mathbb{C}X)$. It is a Lie algebra under the bracket, $[A, B] = AB - BA$.
- When G acts on V elements of $\mathcal{G}$ act as differential operators.
- The exponential map is a diffeomorphism from $\mathcal{G} \to G$, $A \mapsto e^{tA}$ in a neighbourhood of Id.
- The stabilizers K, H are Zariski-closed subgroups of $GL(\mathbb{C}X)$ and they are submanifolds of $GL(\mathbb{C}X)$. Their Lie algebras, $\mathcal{K}, \mathcal{H}$ are complex subspaces of $End(\mathbb{C}X)$, and coincide with the tangent spaces at Id to K, H respectively.
- If H is the stabilizer of a form f, differential operators in $\mathcal{H}$ send f to 0.
- G acts on $\mathcal{G}$, $G \to GL(\mathcal{G}), g \mapsto [\mathfrak{g} \to g\mathfrak{g}g^{-1}]$.

Restricting the above action to $\lambda(t) \subseteq G$,

$$\lambda(t)\mathfrak{g} = \sum_a t^a \mathfrak{g}_a$$

- Can talk of leading terms of every element in $\mathcal{K}$.

Example

- $X = \{x, y, z\}$, $f = (x^2 + y^2 + z^2)^2 \in Sym^4(X)$.
- The stabilizer algebra $\mathcal{K}$ is given below.
- $$\mathcal{K} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$
- $$ay\frac{\partial}{\partial x} + bz\frac{\partial}{\partial x} - ax\frac{\partial}{\partial y} + cz\frac{\partial}{\partial y} - bx\frac{\partial}{\partial z} - cy\frac{\partial}{\partial z}$$
- $\lambda(t) \subseteq GL(X)$ given by $\lambda(x) = x, \lambda(y) = y$ and $\lambda(z) = tz$, as shown below.

$$\lambda(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{bmatrix}$$

- $g = \hat{f} = LT((x^2 + y^2 + t^2z^2)^2) = (x_2 + y^2)^2$. The stabilizer $\mathcal{H}$ is as shown below:

$$\mathcal{H} = \begin{bmatrix} 0 & a & c_1 \\ -a & 0 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$$

- $\hat{K} = LT(\lambda, \mathcal{K})$ is given by the leading terms of

$$\lambda(t)\mathcal{K}\lambda(t)^{-1} = \begin{bmatrix} 0 & a & t^{-1}b \\ -a & 0 & t^{-1}c \\ -tb & -tc & 0 \end{bmatrix}$$

- This is the Lie algebra of matrices with entries

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathcal{H}$$

Let $T_z O(z) = \mathcal{G} \cdot z$, the tangent space to the orbit $O(z) = G \cdot z$ at the point z. Then $V/(T_z O(z))$ is an $\mathcal{H}$-module. We call this the $\star$-action. Let $\mathcal{H}_{y_e}$ be the stabilizer in $\mathcal{H}$ of $\overline{y_e} \in V/(T_z O(z))$.

Proposition

Let $z = \hat{y}$ and $\mathcal{H} = Lie(H)$ and $\mathcal{K} = Lie(K)$, where H, K are as above. Then

- *i* $\hat{\mathcal{K}} \subseteq \mathcal{H}$, *thereby connecting* $\mathcal{K}, \mathcal{H}$.
- *ii Let* $y = y_d + y_e + \sum_{i>e} y_i$ *be the decomposition of* y *by degrees, with* $z = y_d$ *and* y_e *as the tangent of approach. Let* $\mathfrak{k} \in \mathcal{K}$ *be given by* $\mathfrak{k}_a + \mathfrak{k}_{a+1} \ldots$. *Then* $\mathfrak{k}_a, \ldots, \mathfrak{k}_{a+e-d-1} \in \mathcal{H}$ *and* $\mathfrak{k}_a \in \mathcal{H}_{y_e}$. *So,* $\hat{\mathcal{K}} \subseteq \mathcal{H}$. *Moreover,* $\mathfrak{k}_a \cdot \overline{y_e} = 0$, *so* $\hat{\mathcal{K}} \subseteq \mathcal{H}_{y_e} \subseteq \mathcal{H}$.

- The two boundary components resolved by Hüttenhain in his thesis.
- $$Q_1(X) \quad = \quad det\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_5 - x_1 \end{bmatrix}\right)$$
- $$det_3(X) = Q_1(X) + (x_1 + x_5 + x_9)(x_1x_5 - x_2x_4)$$
- Set $Y = \{x_1, \ldots, x_8\}$ and $Z = \{x_1 + x_5 + x_9\}$.
- $\lambda^1(t) \in GL(X)$ as $\lambda^1(t)x_i = x_i$ for $i = 1, \ldots, 8$ and $\lambda^1(t)(z) = tz$, where $z = (x_1 + x_5 + x_9)$.
- $$\lambda^1(t) \cdot det_3(X) = Q_1 + t \cdot Q_1'$$
- $d = 0, e = 1$, the limit $z^1 = Q_1$, the tangent of approach $y_e := Q_1'$.
- $\mathcal{H}_1 = \ell^1 \oplus \hat{\mathcal{K}}^1$, where $t^{\ell^1} = \lambda^1(t)$ and $\hat{\mathcal{K}}^1$ is the leading term algebra.
- $\hat{\mathcal{K}}^1 = \mathcal{H}^1_{y_e}$, the stabilizer of the tangent of approach, and $[\ell^1, \hat{\mathcal{K}}^1] = \hat{\mathcal{K}}^1$.

- $$Q_2(X) = x_4x_1^2 + x_5x_2^2 + x_6x_3^2 + x_7x_1x_2 + x_8x_2x_3 + x_9x_1x_3$$
-

Lemma (Hüettenhain)

Let Y, Z be the generic matrices below and let $X = Y \oplus Z$.

$$Y = \begin{bmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & x_3 \\ x_2 & -x_3 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 2x_6 & x_8 & x_9 \\ x_8 & 2x_5 & x_7 \\ x_9 & x_7 & 2x_4 \end{bmatrix}$$

Let $\lambda^2(t)$ be such that $\lambda^2(t) \cdot Y = Y$ and $\lambda_2(t) \cdot Z = tZ$. Let us define $det^3(X)$ as the determinant of the matrix $Y + Z$. Then:

$$det^3(\lambda^2(t) \cdot X)) = det(Y + tZ) = tQ_2 + t^3 Q_3$$

where:

$$\begin{aligned} Q_2(X) &= x_4x_1^2 + x_5x_2^2 + x_6x_3^2 + x_7x_1x_2 + x_8x_2x_3 + x_9x_1x_3 \\ Q_2'(X) &= 8x_4x_5x_6 - 2x_6x_7^2 - 2x_4x_8^2 - 2x_5x_9^2 + 2x_7x_8x_9 \end{aligned}$$

The 3 x 3 Determinant case...

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- $z^2 = Q_2$ is the limit, $d = 1$ and $e = 3$. $y_e := Q_2'$ is the tangent of approach
- $\mathcal{H}_2 = \ell^2 \oplus \hat{\mathcal{K}}^2$, where $t^{\ell^2} = \lambda^2(t)$ and $\hat{\mathcal{K}}^2$ is the leading term algebra of $\mathcal{K}$ under $\lambda^2(t)$.
- $\hat{\mathcal{K}}^2 = \mathcal{H}^3_{y_e}$, the stabilizer of the tangent of approach, and $[\ell^2, \hat{\mathcal{K}}^2] = \hat{\mathcal{K}}^2$.

The 3 x 3 Determinant case...

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- $\hat{\mathcal{K}}^2 = \mathcal{H}^3_{y_e}$, the stabilizer of the tangent of approach, and $[\ell^2, \hat{\mathcal{K}}^2] = \hat{\mathcal{K}}^2$.

- Recipe to get hold of λ_1, λ_2?
- $\hat{\mathcal{K}}^1$ is obtained via the injection $SL_3 \to SL_3 \times SL_3$, $A \to A \times A^{-1}$.
- The reductive part of $\hat{\mathcal{K}}^1$ is the sl_3-module $\mathbb{C}^8 \oplus \mathbb{C}^1$, corresponding the break-up of $X = X' \oplus cI$, the trace zero matrices X' and the identity matrix.
- $\lambda_1(t)$ commutes with the reductive part!
- $\hat{\mathcal{K}}^2$ is obtained via the injection $SL_3 \to SL_3 \times SL_3$, $A \to A \times A^T$.
- The reductive part of $\hat{\mathcal{K}}^2$ is the diagonal embedding of sl_3 via $(Sym^2(\mathbb{C}^3))^* \oplus Sym^1(\mathbb{C}^3)$ corresponding to the break-up of X as symmetric and antisymmetric matrices.
- $\lambda_2(t)$ commutes with the reductive part!

Intermediate G-stable varieties

Sandwich varieties

- I_z ideal of z in $\mathbb{C}[V]$, I_y ideal of y in $\mathbb{C}[V]$. Both are G-stable and $I_y \subseteq I_z$.
- Use the direction of approach, y_e to z, to construct (suitable)-derivations - directional derivatives in the direction gy_e at gz for every $g \in G$.
- The first thickening is:

$$J^1(\lambda) = \{f \in I_z | D^1_{gz,gy_e}(f) = 0 \text{ for all } g \in G\}$$

- The higher thickenings are

$$J^k(\lambda) = \{f \in I^k_z | D^1_{gz,gy_e}(f) = 0 \text{ for all } g \in G\}$$

- This construction depends only on z and the representative of y_e in $T_z(V)/T_z(O_z)$.
- Set $R_i = I^i_z/I^{i+1}_z$, $R_z = \oplus R_i$. Can be used to get a filtration of $J = \oplus_i J^i$, $\overline{J}$.
- Get a G-map from $R_z/\overline{J} \to \mathbb{C}[G]^{H_{y_e}}$ which allows for reasoning about I_z/I_y and a filtration of it.

Sandwich varieties

- $\lambda(t) \cdot y = t^d z + t^e y_e + \ldots + t^D y_D$
- Let $y' = gy$ and $\lambda(t) \cdot y' = t^a y'_a + \ldots + t^b y'_b$.
- Set $Y_d = \{y' = gy | y'_a = 0 \text{ for all } a < d\}$ — those elements in $O(y)$ for which $deg(\hat{y}') \geq d$.
- Let V_d be the degree d subspace of V under λ, and consider the projection $\pi_d : V \to V_d$.
- Let $Z_d(\lambda) = \pi_d(Y_d)$.
- Every $z' \in Z_d$ is in $O(y')$, so $\overline{GZ_d} \subset \overline{O(y)}$, thus constraining possible y. There is a natural lower bound on the codimension of $O(z)$ in Z_d - based on $\mathcal{H}/\mathcal{H}_{y_e}$, $\mathcal{H}_{y_e}/\hat{\mathcal{K}}$.

Conclusions:

- New proof of why the no occurrence obstructions needs refinement.
- 1-PS subgroups commuting with of large subgroups of K give give us degree 0 components of $\hat{\mathcal{K}}$ (reductive subalgebras) which go into $\mathcal{H}$.
- The thickening varieties allow us to reason about the filtration I_y/I_z. Modules in the coordinate ring of y which are not related to H are in the kernel $\overline{J}/I_y$. Modules are related to H, H_{y_e}.
- Construction of the variety $Z(d)$ which constrains possible y.