On the Universality of border width-2 ABPs over characteristic 2

Joint work with Pranjal Dutta, Balagopal Komarath, Harshil Mittal, and Saraswati Nanoti.

Dhara Thakkar
Indian Institute of Technology Gandhinagar, India.

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Workshop on Algebraic Complexity Theory 2023

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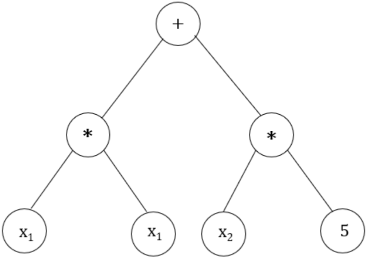
Arithmetic Formula



$x_1{}^2 + 5x_2$

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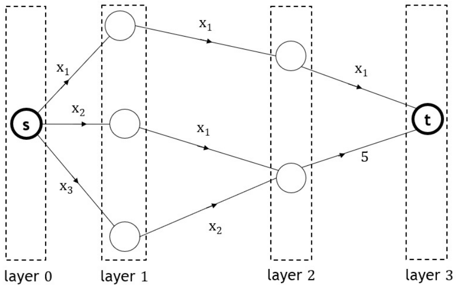
Algebraic Branching Programs (ABP)



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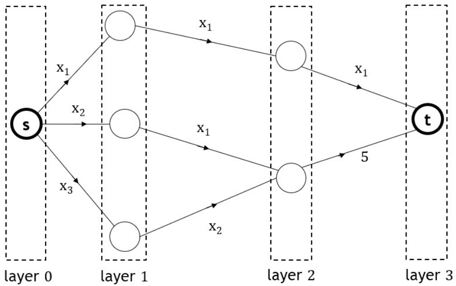

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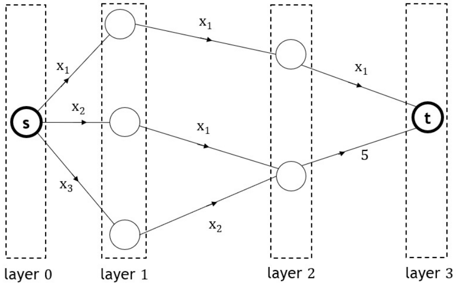

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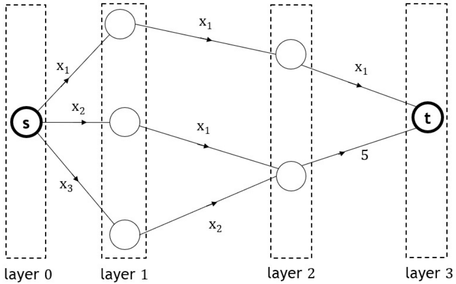

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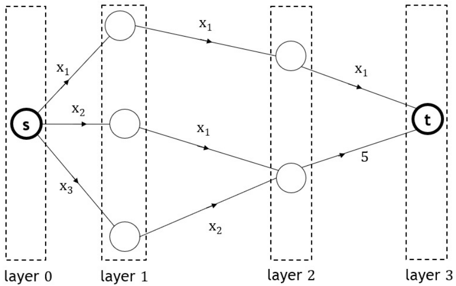

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- We use Weakest ABP.

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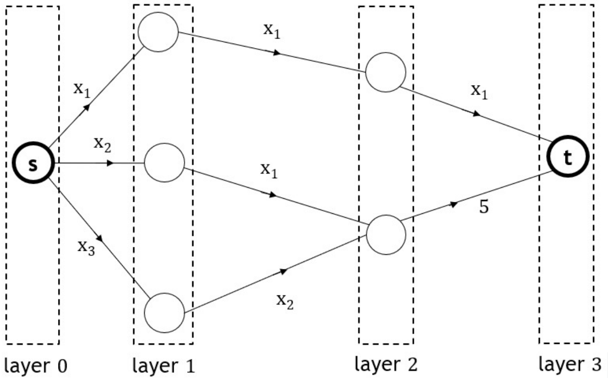
The *width* of an ABP is the maximum number of nodes in a layer.



Here, the width is 3.

ABPs and Matrix Multiplication

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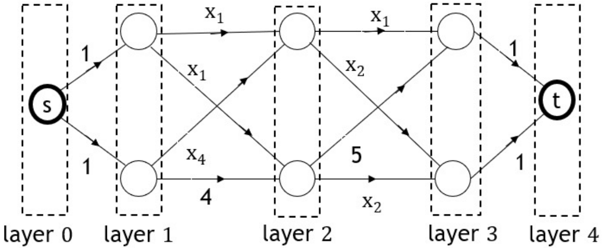

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_1 \\ x_4 & 4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ 5 & x_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This computes the polynomial $x_1^2 + 2x_1x_2 + x_2x_4 + x_1x_4 + 5x_1 + 4x_2 + 20$.

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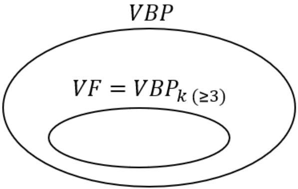
  $\text{VBP}_k =$ VF [Ben-Or and Cleve, SIAM J. Comp., 1992].



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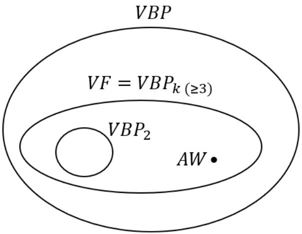

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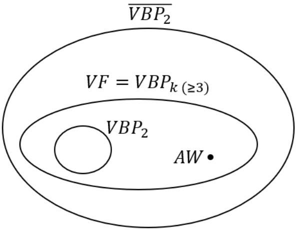
  Then, they become 'at least as powerful' as formulas[1].



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Definition: A polynomial $g \in \mathbb{F}(\epsilon)[X]$ *approximates* $f \in \mathbb{F}[X]$ with *error degree* e if $g = f + \epsilon h_1 + \epsilon^2 h_2 + \cdots + \epsilon^e h_e$ where each $h_i \in \mathbb{F}[X]$.

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- Is $\mathrm{AW} \in \overline{\mathrm{VBP}_2}$?

Approximation Helps

The polynomial $\mathrm{AW} = \sum_{i=1}^{8} x_{2i-1}x_{2i}$ can be approximated by width-2 ABP when $\mathrm{char}(\mathbb{F}) \neq 2$ [BIZ, J.ACM 2018].

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Note: This is true for arbitrary field.

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Thus, $\overline{\mathsf{VF}} = \overline{\mathsf{VBP}_2}$ when $\mathrm{char}(\mathbb{F}) \neq 2$

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Lemma 1

Let f be a polynomial. Suppose that there is a sequence, say σ, of N matrices that approximately computes $Q(f)$. Then, for any indeterminate x, there is a sequence of $2N+4$ matrices that approximately computes $Q(fx)$.

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Let f and g be polynomials. Suppose that there are sequences, say σ and π, of N and M matrices, that approximately compute $Q(f)$ and $Q(g)$ respectively. Then, there is a sequence of $N + 2M + 4$ matrices that approximately computes $Q(fg^2)$.

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Let p be a polynomial with ℓ monomials, each containing at most t indeterminates. Then, $Q(p)$ can be approximately computed using a sequence of at most $\mathcal{O}(\ell \cdot (2^t + \mathit{degree}(p)))$ matrices.

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Observe that x^n needs only $O(n)$ matrices.

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Sequence of $\mathcal{O}(\ell \cdot (2^t + degree(p)))$ matrices that approximately computes $Q(p)$ where,
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Corollary: Let p be a univariate polynomial of degree d. Then $Q(p)$ can be approximately computed using a sequence of at most $\mathcal{O}(d^2)$ matrices.

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Note: However, we can do better!

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Consider a degree-d univariate polynomial in x, say $p = a_d x^d + a_{d-1} x^{d-1} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$. Then, $Q(p)$ can be approximately computed using a sequence of at most $\mathcal{O}(d)$ matrices.

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$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = (a_4x^2 + a_3x + a_2)x^2 + a_1x + a_0.$$

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Observation: There is an n-variate polynomial over $\mathbb{F}$ with $2^{\Omega(n)}$ monomials that can be approximately computed using a sequence of $n^{\mathcal{O}(1)}$ matrices, when $\mathrm{char}(\mathbb{F}) = 2$.

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